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M.Sc. [Mathematics]

I - Semester

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TOPOLOGY - I

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SYLLABI-BOOK MAPPING TABLE

Topology - I

Syllabi	Mapping in Book
BLOCK I: FUNDAMENTALS, FINITE AND INFINITE SETS **UNIT - I** Set Theory and Logic : Fundamental Concepts- Functions- Relations - The Integers and the Real Numbers **UNIT - II** The Integers and the Real Numbers - Cartesian Products- Finite Sets - Countable and Uncountable sets **UNIT - III** Infinite sets and the Axiom of choice- ordered sets- The Maximum Principle	**Unit 1:** Set Theory **(Pages 1-45);** **Unit 2:** Set Theory and Number System **(Pages 46-69);** **Unit 3:** Infinite Sets **(Pages 70-81)**
BLOCK II: TOPOLOGICAL SPACES **UNIT - IV** Topological Spaces - Basis of a topology - Problems **UNIT - V** The order topology - The product topology on X × Y - Problems **UNIT - VI** The subspace topology - Closed sets and limit points - Hausdorff spaces **UNIT - VII** Continuous Functions - Continuity of a Function- Homomorphisms The product topology **UNIT - VIII** Constructing continuous Functions -The metric topology - The quotient topology.	**Unit 4:** Topological Spaces: An Introduction **(Pages 82-93);** **Unit 5:** Order Topology **(Pages 94-107);** **Unit 6:** Topology: Subspace, Closed Sets and Hausdorff Spaces **(Pages 108-134);** **Unit 7:** Continuous Function, Product Topology and Homeomorphism **(Pages 135-150);** **Unit 8:** Metric and Quotient Topology **(Pages 151-158)**
BLOCK III: CONNECTED AND COMPACT SPACES **UNIT - IX** Connected spaces - Connected sets in the real line - component and path components **UNIT - X** Local connectedness - Compact spaces - Problems **UNIT - XI** Compact sets in the real line - Limit point compactness - Local Compactness	**Unit 9:** Connected Spaces and Sets **(Pages 159-170);** **Unit 10:** Local Connectedness and Compact Spaces **(Pages 171-180);** **Unit 11:** Compact Set and Limit Point Compactness **(Pages 181-209)**
BLOCK IV: COUNTABILITY AXIOMS AND NORMAL SPACES **UNIT - XII** The Countability axioms - The separation axioms - Problems **UNIT - XIII** Normal spaces - The Urysohn's lemma **UNIT - XIV** The Uryshon's metrization theorem - Related Problems	**Unit 12:** Countability and Separation Axioms **(Pages 210-227);** **Unit 13:** Normal Spaces **(Pages 228-245);** **Unit 14:** The Uryshon's Metrization Theorem **(Pages 246-254)**

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INTRODUCTION

Topology can be defined as the study of qualitative properties of certain objects, known as *topological spaces*, which are invariant under certain kind of transformations, particularly those properties that are invariant under homeomorphism. Topology is a major area of mathematics concerned with properties that are preserved under continuous deformations of objects, such as deformations that involve stretching. It emerged through the development of concepts from geometry and set theory, such as space, dimension and transformation. The term topology is also used to refer to a structure imposed upon a set X, a structure that essentially distinguishes the set X as a topological space by considering properties, such as convergence, connectedness and continuity, upon transformation. Topological spaces appear unsurprisingly in almost every branch of mathematics, hence this has made topology one of the great unifying concepts of mathematics. The motivating insight behind topology is that some geometric problems depend not on the exact shape of the objects involved, but rather on the way they are put together.

The theory of topological spaces provides a setting for the notions of continuity and convergence which is more general than that provided by the theory of metric spaces. In the theory of metric spaces we can find the necessary and sufficient conditions for convergence and continuity that do not refer explicitly to the distance function on a metric space but instead are expressed in terms of open sets. We can generalize the notions of convergence and continuity by introducing the concept of a topological space as: a topological space consists of a set together with a collection of subsets called open sets that satisfy appropriate axioms. The axioms for open sets in a topological space are satisfied by the open sets in any metric space. The subject of topology itself consists of several different branches, such as point set topology, algebraic topology and differential topology.

This book, *Topology - I*, is divided into four blocks, which are further divided into fourteen units which will help you understand how to distinguish spaces by means of simple topological invariants (compactness, connectedness and the fundamental groups). The topics discussed are set theory and logic, functions, relations, integers and the real numbers, Cartesian products, finite sets, countable and uncountable sets, infinite sets and the axiom of choice, topological spaces, order topology, product topology on $X \times Y$, subspace topology, closed sets and limit points, Hausdorff spaces, continuous functions, homomorphisms, constructing continuous functions, metric topology, quotient topology, connected spaces, path components, local connectedness, compact spaces, local compactness, countability axioms, separation axioms, normal spaces, the Urysohn's lemma and metrization theorem.

The book follows the self-instruction mode or the SIM format wherein each unit begins with an 'Introduction' to the topic followed by an outline of the 'Objectives'. The content is presented in a simple and structured form interspersed with 'Check Your Progress' questions and answers for better understanding. A list of 'Key Words' along with a 'Summary' and a set of 'Self Assessment Questions and Exercises' is provided at the end of the each unit for effective recapitulation.

BLOCK - I

FUNDAMENTALS, FINITE AND INFINITE SETS

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UNIT 1 SET THEORY

Structure

NOTES

1.0 INTRODUCTION

The origins of set theory can be traced to late 19th century when a German mathematician Georg Cantor (1845-1918) published a paper on, "Property of the Collection of All Real Algebraic Numbers". Sets and set theory are regarded as the foundation of mathematics. In terms of mathematics, a set refers to a collection of distinct objects and considered as an object in its own right. The constituent members of a set are known as elements of the set. You will understand different types of set notation forms, such as Roster form and set builder form. Sets are of various types, such as finite and infinite sets, universal sets, disjoint sets, overlapping sets, equivalent sets, empty and null sets, and subset of sets, etc. This unit will introduce you to the various applications of set theory.

You will be acquainted with different types of operators, such as cardinality of a set, union of sets, intersection of sets, difference of two sets, complement of set, set identities, Venn diagram, minset and maxset. Properties of set operations, such as commutative, associative, identity, distributive, laws of idempotence, De Morgan's law, Cartesian product of sets are also discussed.

In mathematics, a relation is used to describe certain properties of things, i.e., a relation is a relationship between sets of values. You will understand various types of relations, such as reflexive, symmetric and transitive relations, and equivalence class and partitions, etc. You will learn about range of function, composition of functions and types of functions, such as one-to-one, onto surjective, bijective, constant, into, identity, inverse, even and odd.

The unit also discusses about mathematical logic, logical operators, the equivalence formula and the respective logical operations for conjunction, disjunction, negation using the conditional operator, bi-conditional operator, etc.

1.1 OBJECTIVES

After studying this unit, you will be able to:

- Know the basics of set theory and features of sets
- Understand various notations in set theory and set elements
- Comprehend set operations and properties of set operators
- Prove and understand DeMorgan's laws
- Define relations with respect to set theory
- Analyse partitions and equivalence class
- Describe the features of various types of functions
- Interpret mathematical logic and use different logical operators
- Understand the special type of relations called functions

1.2 DEFINITIONS AND NOTATIONS

Even though formal definitions have been given for a set and elements of a set, but the meaning of a set and its elements is best comprehended intuitively. A set is a collection or compilation of distinct entities or objects. The objects which form the set are called the elements of the set. For example Collection of students of your class is a set, collection of all rational numbers is a set, and collection of all whole numbers between 10 and 20 is a set. You are an element of the set of students of your class; the city you reside in is an element of the set of all cities in India. A set can be an element of another set, for example a set of girls studying in a class is an element of the set of students studying in that class.

Sets are denoted by capital letters, for example A, B, C……. and elements of a set are denoted by small letters; a, b, c……. If 'A' is a set and 'b' is an element of this set then we write:-

b ∈ A (and read it as b belongs to A or b is an element of A)

If 'b' is not an element of 'A' then we denote it as:

b ∉ A (and read it as b does not belong to A or is not an element of A)

Example 1: Show that $A = \{2, 4, 6, 8\}$ is a set of the even numbers.

Solution: $A = \{2, 4, 6, 8\}$ is a set of the even numbers 2, 4, 6, 8. You may say A is a set of all even numbers between 1 and 9; or 2, 4, 6, 8 are elements of A.

Example 1 gives the *Roster* or *tabulation method* to describe a set. It consists of listing of each element of the set within the braces. Another method is the *descriptive phrase method* which consists of placing a phrase describing the elements of the set within the braces. Thus,

$$A = \{\text{Even numbers between 1 and 9}\}$$

This method may be used when there is a large number of elements or when all the elements cannot be named.

A third method, known as the *rule method* or *set builder* method, consists of enclosing within the brace a general element and describing it. Thus:

$$A = \{x: x \text{ is an even integer and } 1 < x < 9\}$$

Or,

$$A = \{x/x \text{ is even integer and } 1 < x < 9\}$$

Here, : or / is used after x.

A set is determined by its elements and not by the method of description. The set A described in any of these three different ways remains the same.

The idea of a set is both obvious and ancient. But what looks obvious must be proved and the proof may be either insufficient to justify the obvious or it may lead to other possibilities including the opposite of the obvious.

NOTES

The algebra of sets provides a student with laws which can be used to prove important results in any science.

1.2.1 Roster Form

The standard notation to list the elements of a set and denote the set is to use braces. For example set A having elements x, y & z is written as:

A = {a,b,c}, similarly;

B = {0, 1, 2, 3........} is set of whole numbers;

N = { 1, 2, 3, 4, 5} is a set of first 5 natural numbers.

X = { a, b, Y} is a set with elements a, b and the set Y. In this example Set Y is an element of set X.

The above way of enumerating a set is called as Roster form. Some standard sets we have studied earlier are:

Z = Set of integers

Q = Set of rational numbers

R = Set of real numbers.

N = Set of natural numbers

1.2.2 Set Builder Notation

Another way of describing a set is called the Set Builder Notation. In the examples above every set has been denoted by writing its elements. However, it may not be always possible to write all the elements of a set. In such cases the set can be described by the property of its elements. The notation is called Set Builder Notation. It has three parts, a variable, a separator and a rule or function that the variable satisfies. For example:

$A = \{x \mid f(x)\}$

Or

$A = \{x : f(x)\}$

It implies that set A is a set of all elements x which are true for the rule or function f(x). It is read as A is a set of all x such that x is part of f(x). Sometimes a domain is also specified, for example:

$A = \{ x \in N \mid x = 2k \text{ for all } k \in N\}$

1.2.3 Representation of Sets

Sets can be represented by circular or similarly enclosed curves commonly known as Venn diagrams. Every element in a set is represented by a point.

Example 2: With the help of Venn diagram show that,

$$V = \{a, e, i, o, u\}.$$

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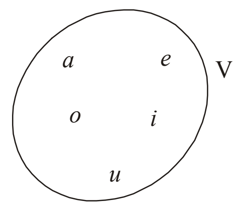
Solution: The Venn diagram shows how the vowels which are the elements of the set V belong to the set V.



You shall use symbols and diagrams to represent and understand sets.

It should be remembered that a set remains unchanged if the order of its elements is changed. Thus, you can write:

$$V = \{e, o, i, a, u\}.$$

Example 3: Show that $B = \{1, 2, 3, 4, 5\}$ is a set of numbers less than 6.

Solution: $B = \{1, 2, 3, 4, 5\}$ is the set of counting numbers less than 6. Alternatively, $B = \{x \text{ (integer)} \mid 0 < x < 6\}$.

Example 4: What does the set C = {Physics, Chemistry, Sociology, Economics} mean?

Solution: C = {Physics, Chemistry, Sociology, Economics} is a set of four sciences.

Example 5: How will you describe the set of the two smallest integers larger than 15?

Solution: $\{16, 17\}$

Example 6: How will you express the following in the set notation?

(i) The letters in the word 'internal'.

(ii) Integers greater than 100 but less than 102.

(iii) Integers less than –5 and greater than or equal to –10.

(iv) Letters of the English alphabet.

(v) Odd numbers upto 99.

Solution: (i) $\{i, n, t, e, r, a, l\}$; (ii) $\{101\}$; (iii) $\{-6, -7, -8, -9, -10\}$; (iv) $\{a, b, c, ..., z\}$; (v) $\{1, 3, 5, ..., 99\}$

These sets may be expressed in different forms.

Example 7: What do the following sets mean?

(i) $A = \{x : x \text{ is a person weighing more than 60 kg}\}$

(ii) $B = \{b : b > 2\}$

(iii) $C = \{c \text{ (integer)} : c < 0\}$.

NOTES

Solution: (i) A consists of the elements x such that x is a person weighing more than 60 kg.

(ii) B is the set of all numbers greater than 2.

(iii) C is the set of all integers less than 0, i.e., C is the set of all negative integers.

1.2.4 Empty Set or Null Set

A set that has no elements is called a null set or an empty set. The concept of an empty set needs to be understood since someone may argue that if a set has no elements then how is it a set. The concept is analogous to that of zero in arithmetic. It becomes important in many situations and is used for defining many properties. For example if you were interested in making a set of all senior citizens living in a society who have their children working in USA, you may have some elements or you may not have any. Therefore, there is a possibility that this set may be an empty set. The terms null set is more commonly used in Measure theory and empty set is generally used in set theory.

An empty set is denoted as

$A = \{\}$ or,

$A = \Phi$

It needs to be clarified here that the following are not equal:

Φ, 0 and $\{0\}$

The first is an empty set, second is an element and the third is a set with 0(zero) as an element.

1.2.5 Equality of Sets

Two sets are said to be equal if they have the same elements. It may be noted that the order of elements in a set does not matter.

Example 8 : If $A = \{a, b, c, d, 1, 2, 3\}$ and

$B = \{1, a, 2, b, 3, d, c\}$, then $A = B$.

Example 9: If $N = \{1, 2, 3 ....\}$ and

$X = \{x \mid x \text{ is a non negative real number}\}$,

then $N \neq X$ because 0 is an element of X but not of N.

1.2.6 Finite and Infinite Sets

A set is finite, if it has a finite number of elements. The elements of such a set can be counted by a finite number. The number of elements in a finite set A is denoted by $n(A)$. Here n is a finite positive integer.

If a set has an infinite number of elements it is an infinite set. The elements of such a set cannot be counted by a finite number.

A set of points along a line or in a plane is called a point set. A finite set has a finite subset. An infinite set may have an infinite subset.

For example, $F = \{a, b, x, z, 0, 18, p\}$. Where F is a finite set. $n(A) = 7$.

Similarly, $S = \{x : x$ is a grain of sand on the sea shore$\}$. Where S is a finite set.

In case, $R = \{x : x$ is a natural number$\}$

$= \{1, 2, 3,...\}$. Then, R is an infinite set.

If, $A = \{x : x$ is a number between 2 and 3$\}$. Then, A is an infinite set.

Example 10: Is $D = \{x : x$ is a multiple of 2$\}$ an infinite set?

Solution: Yes.

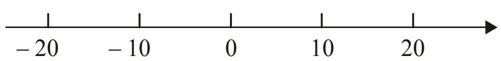
Example 11: Do the letters of the English alphabet give an infinite set?

Solution: No.

Example 12: Is $m = \{x : x$ is a currency note in India$\}$ an infinite set?

Solution: No.

The set of points between 10 and 20 is a point set. There are infinite number of points between 10 and 20. The following is an example of an infinite set.



The set of all points along a line is an infinite set. There are infinite number of points on a line. A member of the set of points along a line can be represented on the line by a variable x.

Example 13: A point set in a plane is shown by the set of points in a plane figure. Is it an infinite set?

Solution: Yes, it is an infinite set.

1.2.7 Universal Set

The universal set is the totality of elements under consideration as elements of any set. It is the set of all objects relevant to a particular application. A universal set is denoted by E. E is the universe or universal set including all possible elements and is often drawn as a rectangle.

For example, $E = \{a, b, c,...z\}$, the set of letters of the English alphabet is a universal set.

$E = \{$head, tail$\}$ is a universal set obtained by the throw of a coin.

$E = \{$rise in demand, constant demand, fall in demand$\}$ is a universal set resulting from a change in the price of a commodity. It covers all the possibilities.

NOTES

1.2.8 Disjoint Sets

If two sets have no element in common, then they are disjoint sets.

If, $A = \{\text{all odd numbers}\}$

And, $B = \{\text{all even numbers}\}$

Then, A, B are termed as disjoint sets.

If $C = \{0, 1, 2, 3, 4\}$

And, $D = \{5, 6\}$

Then, C and D are also termed as disjoint sets. The given Venn diagram illustrates this fact.

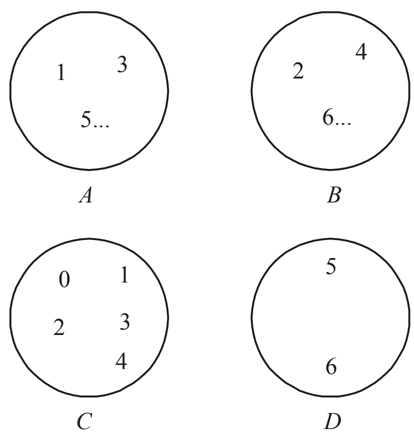


Fig. 1.1 *Disjoint Sets*

1.2.9 Overlapping Sets

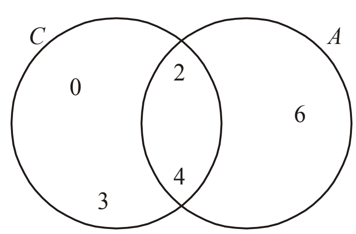
When two sets overlap, the overlapping portion will include the common points between the two sets.

Example 14: With the help of Venn diagram show the common points in overlapping portion of the given sets:

Where, $C = \{0, 2, 3, 4\}$

And, $A = \{2, 4, 6\}$

Solution: The following Venn diagram shows the common points 2, 4 in the overlapping position.



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1.2.10 Membership of a Set

If an element x belongs to a set A, then it is written as $x \in A$.

Read as, x is an element of A.

If an element x does not belong to a set A, then it is written as $x \notin A$.

Hence, x is not an element of A.

If an element x belongs to A it does not mean that the set containing x belongs to A.

$x \in X$ does not mean $\{x\} \in X$

$x \in C, x \notin D$ means that x belongs to C but not to D

$x \in P, x \in Q$ means that x belongs to P and also to Q.

$x \notin A, x \notin B$ means that x does not belong to A or to B.

For example, in the set $X = \{x: x^2 - 5x + 6 = 0\}$, $2 \in X, 3 \in X$

$\because$ 2, 3 are solutions of,

$$x^2 - 5x + 6 = 0 \therefore X = \{2, 3\}.$$

Check Your Progress

1. Define a set. What are elements of a set?
2. What do you understand by Roster form?
3. Define null set or empty set.
4. What do you mean by equality of sets?
5. Define a finite set.
6. Define an infinite set.

1.3 SUBSET OF SETS

Set A is called a subset of B if all elements of A are also the elements of B. It is denoted as given below:

$$A \subseteq B$$

In other words it can be said that A = B if and only if $A \subseteq B$ and $B \subseteq A$.

Further, if $A \subseteq B$ but $A \neq B$ then A is called to be a proper subset of B. Written as $A \subset B$. The relations subset and proper subset is also called inclusion and proper inclusion respectively.

$A \subset B$ can also be written as $B \supset A$ and read as B is a superset of A or B contains A.

It may be noted that Φ is a subset of all sets. As also each set is subset of itself.

Example 1 Set A = {2, 4, 6} $\subseteq$ of set of even numbers.

Example 2 If R = {a, b, c, d, e}, and S = {b, d} then S $\subseteq$ R

1.3.1 Subsets

NOTES

If from a set you want to select some elements having a special property, it means you are interested in a subset of the set.

A subset of the set X is a set which consists of some or all of the elements X. The set Y is a subset of X if and only if every element of Y is an element of X and you write $Y \subseteq X$. $a \in Y$ implies $a \in X$.

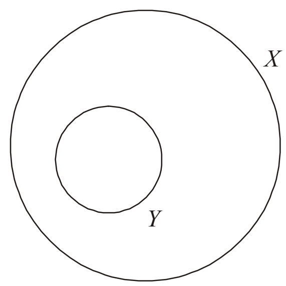

Fig. 1.2 *Subset*

This is called the *inclusion relation*. Every element of Y is also an element of X, i.e., Y is a subset of X or included in X or X is a superset of Y.

A set Y is a *proper subset* of a set X if and only if $Y \subset X$ and $Y \neq X$ and $Y \neq \phi$.

Every set is a subset of itself. Thus, $X \subseteq X$.

The following examples will make the concept clear.

$Y = \{2, 4, 6, 8, 10\}$ is a subset of $X = \{1, 2, 3,..., 10\}$.

$V = \{a, e, i, o, u\}$ is a subset of the set of English alphabets.

$A = \{a, b,..., z\}$.

The set of positive integers is a subset of the set of all real numbers.

If A, B are equal sets, then $A \subset B$ and $B \subset A$.

A finite set has a finite subset. An infinite set may have an infinite subset.

If a set Y is not a subset of a set X, then it is written as:

$$Y \not\subset X$$

Here, the set Y may have some points which are also in X but not all the points of Y are in X.

If a set Y is a subset of set X, all the elements of Y must be in X. A subset can be formed by taking a certain number of elements from the given set.

If, $\quad Y = \{y: 0 \leq y \leq 5\}$

NOTES

And, $X = \{y: 0 \leq y \leq 10\}$

Then, $Y \subset X$

If, $Y = \{a, b, c\}, X = \{a, b, c\}$

Then, $Y \subset X$ and $X \subset Y$

Example 15: State whether the following are true or false.

(i) $E \subseteq E$	(ii) $\phi \subseteq \phi$	(iii) $\phi \in \phi$
(iv) $\phi \subseteq E$	(v) $\phi \subset E$	(vi) $\phi = \{\phi\}$
(vii) $\phi \subseteq \{\phi\}$	(viii) $\phi \subseteq A$	(ix) $\phi = \{0\}$

Solution: (i) T, (ii) T, (iii) F, (iv) T, (v) T, (vi) F, (vii) T, (viii) T, (ix) F

Example 16: If $A = \{a, b, c\}$, is A a subset of A?

Solution: Yes

If A = {Money, Banking, Public Finance}

And, B = {x: x = All subjects in Economics}

Then, $A \subseteq B$.

Also, the empty set is a subset of every set.

Thus, $\phi \subseteq A$ for any set A.

It means all elements of ϕ are elements of A. And ϕ has no elements at all.

The following examples will make the concept clear:

If $A = \{2, 4, 6, 8, 10\}$ and B = {Even integers}, then $A \subseteq B$.

Set $\{1, 2\}$ is a proper subset of $\{1, 2, 3\}$.

1.3.2 Transitivity of Subsets

If X, Y, Z are sets and $X \subseteq Y$ and $Y \subseteq Z$, then $X \subseteq Z$.

Check Your Progress

7. What is a subset?
8. What is a proper subset?
9. Show transitivity of subsets.

1.4 SET OPERATORS

1.4.1 Cardinality of a Set

Cardinality of a set A is the number of elements of set A and is denoted as |A|.

NOTES

1.4.2 Union of Sets

Union of two sets X & Y is said to be the set of all elements of set X along with all elements of set Y. It is denoted as 'U'. X U Y is a set containing all elements that belong to set X, or to set Y or to both.

$$X \cup Y = \{ x | x \in X \text{ or } x \in Y\}$$

Example 17: If A = {2, 4, 7, 8}, B = {5, 7, 8}. Then

$A \cup B = \{2, 4, 5, 7, 8\}$.

It can be seen that for a set X, following is true for union:

(a) $X \cup \varphi = X$ and

(b) $X \cup X = X$

Example 18

$$A \cup B = \{x: x \in A \text{ or } x \in B\}$$
$$A = \{2, 4, 6, 8, 10\} \text{ and } B = \{4, 8, 12\}$$
$$A \cup B = \{2, 4, 6, 8, 10, 12\}$$

1.4.3 Intersection of Sets

Intersection of two sets X and Y is a set of all elements that are common to both X and Y. It is denoted as $\cap$.

$$X \cap Y = \{ x | x \in X \text{ and } x \in Y\}$$

It may be noted that if sets X and Y do not have any elements in common then $X \cap Y = \Phi$.

Example 19: A = {1, f}, B = {1, b, c, d}, C = {a, c, f, 2}.

Then $A \cap B = \{1\}$ and $(A \cap B) \cup C = \{1, a, c, 2, f\}$,

It can be seen that for a set X, following is true for intersection:

(a) $X \cap \varphi = \varphi$ and

(b) $X \cap X = X$

Example 20:

(i) $A = \{x: x$ is an English alphabet$\}$ and $B = \{x: x$ is a vowel in English alphabet$\}$

$A \cap B = \{a, e, i, o, u\}$

(ii) $A = \{1, 2, 3, 4\}, B = \{2, 4\}$

$A \cap B = \{2, 4\}$

(iii) $A = \{a, b, c\}; B = \{1, 2, 3, 4\}$

$A \cap B = \{\}$

1.4.4 Difference of Two Sets

Difference of two sets X & Y is the set of all elements that belong to X but do not belong to Y.

$$X - Y = \{ x | x \in X, x \notin Y\}$$

NOTES

Example 21: If $B = \{1, b, c, d\}$, $C = \{a, c, f, 2\}$.

Then $B - C = \{1, b, d\}$ and $C - B = \{ a, f, 2 \}$,

Example 22:

$$A = \{a, b, c, d, e, f, g\} \text{ and } B = \{a, b, c, d, h, i, j\}$$

$$A - B = \{e, f, g\} \text{ and } B - A = \{h, i, j\} \text{ [Note that } A - B \neq B - A]$$

1.4.5 Complement of a Set

Complement of a set X in the Universal set U is defined as a set of all elements that belong to U but do not belong to X.

Example 23: Let $U = \{a, b, c, d, e, f, g\}$, and $A = \{b, c\}$, $B = \{a, b, c, e\}$, then

$A^c = \{a, d, e, f, g\}$ and $B^c = \{d, f, g\}$

Example 24: $A^c = \{x \mid x \in \mu \text{ and } x \notin A\}$

$$\begin{aligned} \mu &= \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} \\ A &= \{1, 3, 5, 7, 9\}, B = \{2, 4, 6, 8, 10\}, C = \{3, 6, 9\} \\ A^c &= \mu \backslash A = \{2, 4, 6, 8, 10\} \\ B^c &= \mu \backslash B = \{1, 3, 5, 7, 9\} \\ C^c &= \mu \backslash C = \{1, 2, 4, 5, 7, 8, 10\} \end{aligned}$$

Example 25: Let A and B be two sets. Then,

(i) $(A \cap B)^c = A^c \cup B^c$

(ii) $(A \cap B)^c = A^c \cup B^c$

Solution:

(i) Let $x \in (A \cap B)^c$

$$\begin{aligned} &\Rightarrow x \notin A \cap B \\ &\Rightarrow x \notin A \text{ and } x \notin B \\ &\Rightarrow x \in A^c \text{ or } x \in B^c \\ &\Rightarrow x \in A^c \cup B^c \\ &\therefore (A \cap B)^c \subseteq A^c \cup B^c \qquad ...(1) \end{aligned}$$

(ii) Now, let $y \in A^c \cup B^c$

$$\begin{aligned} &\Rightarrow y \in A^c \text{ or } y \in B^c \\ &\Rightarrow y \notin A \text{ and } y \notin B \\ &\Rightarrow y \notin A \cap B \\ &\Rightarrow y \in (A \cap B)^c \\ &\Rightarrow A^c \cap B^c \subseteq (A \cup B)^c \qquad ...(2) \end{aligned}$$

From Equations (1) and (2), $(A \cap B)^c = A^c \cup B^c$.

Similary, we can prove that the other identity $(A \cup B)^c = A^c \cap B^c$.

NOTES

Set Identities

Table 1.1 represents the set identities.

Table 1.1 *Set Identities*

Identity	***Law***
$A \cup \phi = A$, where ϕ = Null set $A \cup \mu = A$, where μ = Universal set	Identity laws
$A \cup \mu = \mu$ $A \cup \phi = \phi$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$(A^c)^c = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup B)$	Distributive laws
$(A \cap B)^c = A^c \cup B^c$ $(A \cup B)^c = A^c \cap B^c$	De Morgan's laws

1.4.6 Venn Diagram

A diagrammatic representation of set operation is called Venn diagram. Now You will learn how to represent every operation of sets as a Venn diagram.

Square or rectangle represents an universal set and circle or ellipse represent sets.

Square or rectangle represents an universal set and circle or ellipse represent sets.

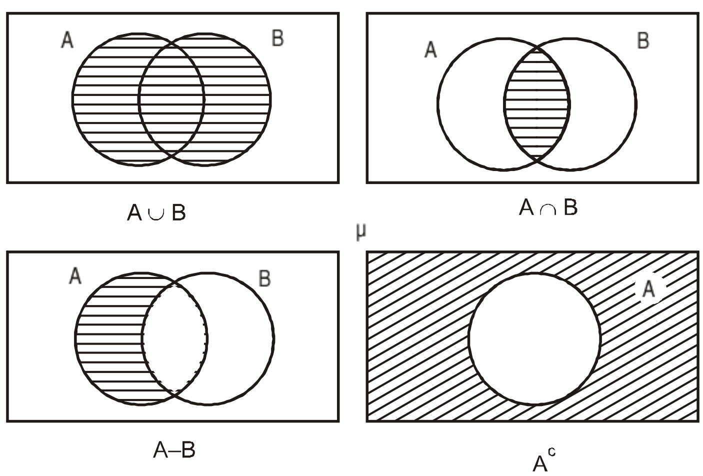

Fig. 1.3 *Set Operation on Venn Diagram*

NOTES

1.4.7 Minset and Maxset

Let U be a universal set and $\{A_1, A_2, \ldots, A_n\}$ be a set of subsets of U. Then, the set of the form $B_1 \cap B_2 \cap \ldots \cap B_n$ where A_i may be either the set B_i or its complement $A^c_{i,}$ is called a *minset* generated by A_i's for all i.

For example, let $A_1 = \{2, 5, 7\}$ and $A_2 = \{2, 4, 5\}$ be two subsets of the set U = {2, 3, 4, 5, 6, 7}. The set A may be partitioned as shown below:

$B_1 = A_1 \cap A'_2 = \{2, 5, 7\} \cap \{3, 6, 7\} = \{7\}$

Similarly,

$$B_2 = A_1 \cap A_2$$
$$= \{2, 5, 7\} \cap \{2, 4, 5\} = \{2, 5\}$$
$$B_3 = A^c_1 \cap A_2 = \{3, 4, 6\} \cap \{2, 4, 5) = \{4\}$$
$$B_4 = A^c_1 \cap A^c_2 = \{3, 4, 6\} \cap \{3, 6, 7\} = \{3, 6\}$$

The following figure shows that none of the sets B_1, B_2, B_3 and B_4 contains common elements and hence the set generated by A_1 and A_2 is the partition of the set U.

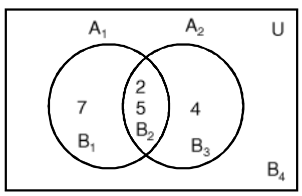

Fig. 1.4 *Minset on Venn Diagram*

In the above Venn diagram the minsets of U are represented.

If union is used instead of intersection a maxset is defined.

$$B_1 = A_1 \cup A^C_2 = \{2, 5, 7\} \cap \{3, 6, 7\} = \{2, 3, 5, 6, 7\}$$
$$B_2 = A_1 \cup A_2 = \{2, 5, 7\} \cup \{2, 4, 5\} = \{2, 4, 5, 7\}$$
$$B_3 = A^C_1 \cup A_2 = \{3, 4, 6\} \cup \{2, 4, 5\} = \{2, 3, 4, 5, 6\}$$

And, $B_4 = A^C_1 \cup A^C_2 = \{3, 4, 6\} \cup \{3, 6, 7\} = \{3, 4, 6, 7\}$

Check Your Progress

10. What is meant by cardinality of a set?
11. What is union of sets?
12. How the two sets can be different?
13. Define complement of a set.
14. What is are disjoint sets?

1.5 PROPERTIES OF SET OPERATIONS

Important laws of sets are listed as follows:

NOTES

1.5.1 Commutative

Union and intersection of sets are commutative. In other words, for any sets x and y

$X \cup Y = Y \cup X$

$X \cap Y = Y \cap X$

1.5.2 Associative

Union and intersection of sets are associative. It implies that, for any three sets x, y, z

$X \cup (Y \cup Z) = (X \cup Y) \cup Z$

$X \cap (Y \cap Z) = (X \cap Y) \cap Z$

1.5.3 Identity

The identity under set union is the empty set Φ and the identity under intersection is the universal set U.

$X \cup \Phi = \Phi \cup X = X$

$Y \cap U = U \cap Y = Y$ (note U denotes universal set)

1.5.4 Distributive

The union operation is distributive over intersection and intersection operation is distributive over union.

$X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$

$X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup TZ)$

1.5.5 DeMorgan's a Law

DeMorgan's laws state that:

(a) $(X \cup Y)^c = X^c \cap Y^c$. (b) $(X \cap Y)^c = X^c \cup Y^c$.

1.5.6 Cartesian Product of Sets

Cartesian product of two sets R and S is the set

$R \times S = \{(r, s) : (r \in R) \text{ and } (s \in S)\}$.

1.5.7 Laws of Idempotence

For any set X,

$X \cup X = X$ and $X \cap X = X$

NOTES

It needs to be noted that the Cartesian product of two sets results into a set. Elements of this set are ordered pairs, the first component of which is an element of the first set (in this case R) and second component is an element of second set (in this case S).

Example 26: If A = {1,2,3} and B = {a, b}, then

$$A \times B = \{(1,a),(1,b),(2,a),(2,b),(3,a),(3,b)\}.$$

Example 27: Let A, B, C be any three sets. Prove that,

$$A \cap (B - C) = (A \cap B) - (A \cap C).$$

Solution: $(A \cap B) - (A \cap C) = (A \cap B) \cap (A \cap C)'$

$= (A \cap B) \cap (A' \cup C')$ (By De Morgan's law)

$= [(A \cap B) \cap A'] \cup [(A \cap B) \cap C']$ (By Distributive law)

$\equiv [(A \cap A') \cap B] \cup [(A \cap B) \cap C']$ (By Associative law)

$\equiv [\phi \cap B] \cup [A \cap (B \cap C')]$

$\equiv \phi \cup [A \cap (B \cap C')]$

$\equiv A \cap (B \cap C')$

$\equiv A \cap (B - C).$

Example 28: For any sets A and B, show that

$$(A - B) \cup (B - A) = (A \cup B) - (A \cap B).$$

Solution: $(A \cup B) - (A \cap B) = (A \cup B) \cap (A \cap B)'$

$= (A \cup B) \cap (A' \cup B')$ (By De Morgan's Law)

$= [(A \cup B) \cap A'] \cup [(A \cup B) \cap B']$ (By Distributive Law)

$= [(A \cap A') \cup (B \cap A')] \cup [(A \cap B') \cup (B \cap B')]$

$= [\phi \cup (B \cap A')] \cup [(A \cap B') \cup \phi]$

$= (B \cap A') \cup (A \cap B')$

$= (B - A) \cup (A - B)$

$= (A - B) \cup (B - A).$ (By Commutative Law)

Check Your Progress

15. What is the identity under set union?
16. State DeMorgan's law.
17. What is the law of idempotence?

1.6 APPLICATIONS OF SET THEORY

Set theory has applications in different branches of mathematics. This is so because set theory is very general; rather abstract in nature. In analysis of mathematical functions, understanding of limit points and concepts of continuity are based on set theory. Boolean algebra can be viewed as algebric treatment of set operations. Intersection, union and difference correspond to logical operations AND, OR and

NOTES

NOT, respectively. These logical operations are switching operations on which entire logic circuits are based.

You also find application of set theory in principle of counting. Sum rule and product rule are expressed in set theoretic notations. When two tasks cannot be performed simultaneously and a task is performed in n_1 ways which is a set A having n_1 elements (ways of doing tasks), and independent of that another task is performed in n_2 ways which is a set containing n_2 elements (way of doing tasks), then both the tasks can be performed in $n_1 + n_2$ ways. Performing both the tasks can be given in set theoretic notation as follows:

$n(A \cup B) = n(A) + n(B)$. Here, $n(A) = n_1$ and $n(B) = n_2$.

Similarly, product rule can also be given a set theoretic presentation. If a task can be performed in n_1 ways, and independent of this task, the second task can be performed in n_2 ways, so that these two tasks, when combined, can be performed in $n_1 n_2$ ways. In set theoretical notation, the rule of product can be interpreted as follows:

$$n(A \times B) = n(A) \times n(B)$$

Where, $n(A)$ $(= n_1)$ and $n(B)$ $(= n_2)$ denotes the number of elements in the sets A and B respectively which denotes ways of performing tasks. Set oriented operations are also found in probability. Relation between elements of one set to another is dealt in defining relations and functions. We know that relation is a subset of Cartesian product of two sets. On the concept of relation design of databases are based. A relational database has concept of Cartesian product in its design. A database is a collection of records, having n-tuples and each tuple has a number of fields. A database of employees records is made up of fields containing name, empnumber, department and salary. Here, employees records are represented as 4-tuples of the form (EMPNAME, EMPNUMBER, DEPT, SALARY).

Consider an example of three such records.

(RAM, P001, PURCHASE, 10000)

(RAJA, M002, MARKETING, 12000)

(ASHA, G001, FRONTOFFICE, 7000)

In database, a table is a relation. An n-ary relation among the sets $A_1, A_2, ..., A_n$ is known as table on these sets that is known as domain of the table. Primary key is given a domain in which two n-tuples in a relation do not have same value and in that case combinations of these domains too, in an n-ary relation, provide unique identification for n-tuples. Cartesian product in domains of such a relation is known as a composite key.

In general, there are many operations on such relations forming new relations. Two important operations (*i*) Join and (*ii*) Projection are used in relational databases. The join operation combines two tables having same domains, i.e., same identical fields. The projection operations form new relations by deletion of same domains in each record of the relation.

The concept of sets finds its use in error detection in the field of computer science. In hamming code this concept is used. This code can be discussed in brief with the help of Venn diagrams. Just to explain the concept you take only 4 bits of data. Figure 1.1 shows the Venn diagrams having data bits, filled in the intersecting inner compartments. Parity bits are filled for these four data bit. You use even parity bit and hence, add parity bits in way that the total number of l's in a circle is even. This is shown in (b) part of Figure 1.1. In the three intersecting circle, each circle have even number of l's.

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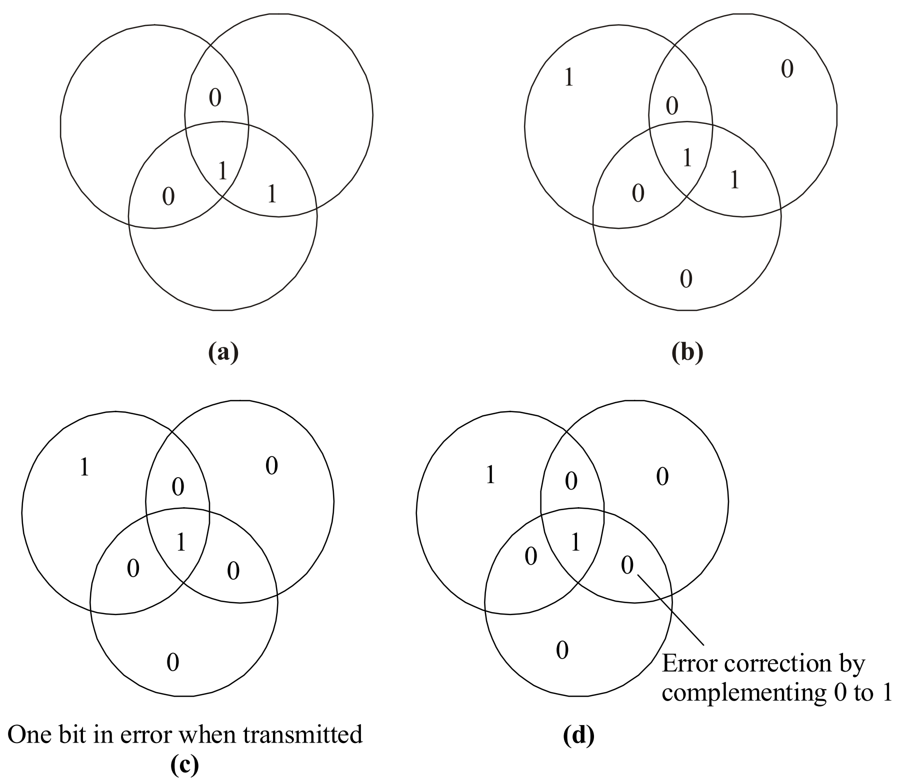

Fig. 1.5 *Venn Diagrams with Data Bits*

When data is transferred an error has occurred which is shown in (c) part of the figure. It is noted that error has occurred in data bit and no other bit has been affected. To detect an error, numbers of 1's are noted. In the figures above, it is noted that there is an error of one bit as shown in the part (d) of the figure and this is corrected. Much complex analysis is performed and these are all based on set theoretic concept.

The following examples will help you understand the applications of set theory.

If A is a finite set, then you shall denote the number of elements in A by $n(A)$. If A and B are two finite sets, then it is very clear from the Venn diagram of $A - B$ that,

$$n(A - B) = n(A) - n(B \cap A)$$

NOTES

Suppose A and B are two finite sets such that $A \cap B = \phi$. Then clearly, the number of elements in $A \cup B$ is the sum of number of elements in A and B the number of elements in B.

i.e., $\quad n(A \cup B) = n(A) + n(B)$ if $A \cap B = \phi$

To find the number of elements in $A \cup B$, in case $A \cap B \neq \phi$, you have to proceed as follows:

You know that for any two sets A and B

$$A \cup B = A \cup (B - A)$$

Here, $A \cap (B - A) = \phi$

Therefore, $n(A \cup B) = n(A) + n(B - A)$

$$= n(A) + n(B) - n(A \cap B)$$

Note: From the definition of empty set, it follows that $n(\phi) = 0$.

So, the following has been proved.

'If A and B are two finite sets, then

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

Similarly, if A, B, C are three finite sets, then

$$\begin{aligned}
n(A \cup B \cup C) &= n(A \cup B) + n(C) - n[(A \cup B) \cap C] \\
&= n(A) + n(B) - n(A \cap B) + n(C) - n[(A \cup B) \cap C] \\
&= n(A) + n(B) + n(C) - n(A \cap B) - n[(A \cap C) \cup (B \cap C)] \\
&= n(A) + n(B) + n(C) - n(A \cap B) - [n(A \cap C) \\
&\qquad + n(B \cap C) - n[(A \cap C) \cap (B \cap C)]] \\
&= n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) \\
&\qquad + n(A \cap B \cap C) \text{ as } A \cap C \cap B \cap C = A \cap B \cap C
\end{aligned}$$

Now you can use these two results in the following problems.

Example 29: In a recent survey of 400 students in a school, 100 were listed as smokers and 150 as chewers of gum; 75 were listed as both smokers and gum chewers. Find out how many students are neither smokers nor gum chewers.

Solution: Let U be the set of students questioned. Let A be the set of smokers, and B the set of gum chewers.

Then, $n(U) = 400$, $n(A) = 100$, $n(B) = 150$, $n(A \cap B) = 75$

We want to find out $n(A' \cap B')$

Now, $\quad A' \cap B' = (A \cup B)' = U - (A \cup B)$

Therefore,

$$\begin{aligned}
n(A' \cap B') &= n[U - (A \cup B)] \\
&= n(U) - n[A \cup B) \cap U] \\
&= n(U) - n(A \cup B) \\
&= n(U) - n(A) - n(B) + n(A \cap B)
\end{aligned}$$

$$= 400 - 100 - 150 + 75$$
$$= 225$$

Example 30: Out of 500 car owners investigated, 400 owned Fiat cars and 200 owned Ambassador cars; 50 owned both Fiat and Ambassador cars. Is this data correct?

NOTES

Solution: Let U be the set of car owners investigated. Let A be the set of those persons who own Fiat cars, B the set of persons who own Ambassador cars; then $A \cap B$ is the set of persons who own both Fiat and Ambassador cars.

$$n(U) = 500,\ n(A) = 400,\ n(B) = 200,\ n(A \cap B) = 50$$

Therefore,
$$\begin{aligned} n(A \cup B) &= n(A) + n(B) - n(A \cap B) \\ &= 400 + 200 - 50 \\ &= 550 \end{aligned}$$

This exceeds the total number of car-owners investigated.

So, the given data is not correct.

Example 31: In a certain government office there are 400 employees. There are 150 men, 276 university graduates, 212 married persons, 94 male university graduates, 151 married university graduates, 119 married men and 72 married male university graduates. Find the number of single women who are not university graduates.

Solution: Let, U = Set of employees

A = Set of men

B = Set of married persons

C = Set of university graduates

Then, $A \cap B$ = Set of married men

$A \cap C$ = Set of male university graduates

$B \cap C$ = Set of married university graduates

$A \cap B \cap C$ = Set of married male university graduates

Now $n(U) = 400$, $n(A) = 150$, $n(B) = 212$, $n(C) = 276$, $n(A \cap B) = 119$, $n(A \cap C) = 94$, $n(B \cap C) = 151$, $n(A \cap B \cap C) = 72$

You have to find out $n(A' \cap B' \cap C')$

Now, $(A' \cap B' \cap C') = (A \cup B \cup C)' = U - (A \cup B \cup C)$

Therefore,

$$\begin{aligned} n(A' \cap B' \cap C') &= n(U) - n[(A \cup B \cup C) \cap U] \\ &= n(U) - n[A \cup B \cup C] \\ &= n(U) - [n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) \\ &\quad - n(B \cap C) + n(A \cap B \cap C)] \\ &= 764 - 710 = 54. \end{aligned}$$

So, the number of single women who are not university graduates is 54.

NOTES

Example 32: A market research group conducted a survey of 1000 consumers and reported that 720 consumers liked product A and 450 consumers liked product B. What is the least number that must have liked both products?

Solution: Let, U = Set of consumers questioned

S = Set of consumers who liked product A

T = Set of consumers who liked product B

Then, $S \cap T$ = Set of consumers who liked both products

Now, $n(U) = 1000,\ n(S) = 720,\ n(T) = 450$

Therefore, $n(S \cup T) = n(S) + n(T) - n(S \cap T)$

$= 1170 - n(S \cap T)$

So, $n(S \cap T) = 1170 - n(S \cup T)$

Now, $n(S \cap T)$ is least when $n(S \cup T)$ is maximum. But $S \cup T \subseteq U$ implies that $n(S \cup T) \leq n(U)$

This implies maximum value of $n(S \cup T)$ is 1000.

So, least value of $n(S \cap T) = 170$.

Hence, the least number of consumers who liked both products is 170.

Example 33: Out of 1000 students who appeared for C.A. Intermediate Examination, 750 failed in Maths, 600 failed in Accounts and 600 failed in Costing, 450 failed in both Maths and Accounts, 400 failed in both Maths and Costing, 150 failed in both Accounts and Costing. The students who failed in all the three subjects were 75. Prove that the above data is not correct.

Solution: Let, U = Set of students who appeared in the examination

A = Set of students who failed in Maths

B = Set of students who failed in Accounts

C = Set of students who failed in Costing

Then, $A \cap B$ = Set of students who failed in Maths and Accounts

$B \cap C$ = Set of students who failed in Accounts and Costing

$A \cap C$ = Set of students who failed in Maths and Costing

$A \cap B \cap C$ = Set of students who failed in all three subjects

Now, $n(U) = 1000,\ n(A) = 750,\ n(B) = 600\ n(C) = 600,\ n(A \cap B) = 450,$

$n(B \cap C) = 150,\ n(A \cap C) = 400,\ n(A \cap B \cap C) = 75.$

Therefore, $n(A \cup B \cup C) = 750 + 600 + 600 - 450 - 150 - 400 + 75$

$= 1025.$

This exceeds the total number of students who appeared in the examination.

Hence, the given data is not correct.

Example 34: In a survey of 100 families, the number that read recent issues of a certain monthly magazine were found to be: September only 18, September but not August 23; September and July 8; September 26; July 48; July and August 8; none of the three months 24. With the help of set theory, find

NOTES

(i) How many read August issue?

(ii) How many read two Consecutive issues?

(iii) How many read the July issue, if and only if they did not read the August issue?

(iv) How many read the September and August issues but not the July issue?

Solution: Let, A = Set of those families that read the September issue

B = Set of those families that read the August issue

C = Set of those families that read the July issue

Then, $A - B$ = Set of those families that read September issue but not August

$A \cap C$ = Set of those families that read both September and July issue

$A \cap B$ = Set of those families that read both September and August issue

$A' \cap B' \cap C'$ = Set of those families that read none

$A - (B \cup C)$ = Set of those families that read September issue only

Now $n(A) = 26$, $n(B) = 8$, $n(C) = 48$, $n(A - B) = 23$, $n(A \cap C) = 8$, $n(B \cap C) = 8$, $n(A' \cap B' \cap C') = 24$, $n[A - (B \cup C)] = 18$

Now, $n(A - B) = n(A) - n(A \cap B)$

So, $23 = 26 - n(A \cap B)$

Therefore, $n(A \cap B) = 3$.

Again, $n[A - (B \cup C)] = n(A) - n[A \cap (B \cup C)]$

So,
$$18 = 26 - n[(A \cap B) \cup (A \cap C)]$$
$$= 26 - n(A \cap B) - n(A \cap C) + n(A \cap B \cap C)$$
$$= 26 - 3 - 8 + n(A \cap B \cap C)$$

Therefore, $n(A \cap B \cap C) = 3$

Also
$$A' \cap B' \cap C' = (A \cup B \cup C)'$$
$$= U - (A \cup B \cup C)$$

Where, U = Set of those families that are questioned and $n(U) = 100$.

So, $n(A' \cap B' \cap C') = n(U) - n(A \cup B \cup C)$
$$24 = 100 - n(A \cup B \cup C)$$

Therefore, $n(A \cup B \cup C) = 76$.

Now,
$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$$

This implies that, $76 = 26 + n(B) + 48 - 3 - 8 - 8 + 3$

So, $n(B) = 18$

This gives the number of families that read the August issue.

Now, $(A \cap B) \cup (B \cap C)$ = Set of those families that read two consecutive issues.

So, $n[(A \cap B) \cup (B \cap C)] = n(A \cap B) + n(B \cap C) - n(A \cap B \cap C)$
$= 3 + 8 - 3 = 8.$

NOTES

Again $C - B$ = Set of those families that read the July issue but not the August issue.

Then, $n(C - B) = n(C) - n(B \cap C)$
$= 48 - 8 = 40.$

Now, $(A \cap B) - C$ = Set of those families that read the September and August issues but not July.

So, $n[(A \cap B) - C] = n(A \cap B) - n(A \cap B \cap C)$
$= 3 - 3 = 0$

Therefore, $(A \cap B) - C = \phi.$

Hence, there is no family that read both the September and August issue but not the July issue.

Example 35: A factory inspector examined the defects in hardness, finish and dimensions of an item. After examining 100 items he gave the following report:

All three defects 5, defects in hardness and finish 10, defects in dimension and finish 8, defects in dimension and hardness 20. Defects in finish 30, in hardness 23, in dimension 50. The inspector was fined. Why?

Solution: Suppose H represents the set of items which have defect in hardness, F represents the set of items which have defect in finish and D represents the set of items which have defect in dimension.

Then, $n(H \cap F \cap D) = 5$, $n(H \cap F) = 10$, $n(D \cap F) = 8$
$n(D \cap H) = 20$, $n(F) = 30$, $n(H) = 23$, $n(D) = 50$

So, $n(H \cup F \cup D) = 30 + 23 + 50 -20 - 10 - 8 + 5 = 70$

Now, $n(D \cup F) = n(D) + n(F) - n(D \cap F)$
$= 50 + 30 - 8 = 72$

$D \cup F \subseteq D \cup F \cup H$ implies $n(D \cup F) \leq n(D \cup F \cup H)$ i.e., $72 \leq 70$.

Hence, there is an error in the report and for this reason inspector was fined.

Example 36: In a survey of 100 families the numbers that read the most recent issues of various magazines were found to be as follows:

Readers Digest	28
Readers Digest and Science Today	8
Science Today	30
Readers Digest and Caravan	10
Caravan	42
Science Today and Caravan	5
All the three Magazines	3

Using set theory, find

(i) How many read none of the three magazines?

(ii) How many read Caravan as their only magazine?

(iii) How many read Science Today if and only if they read Caravan?

Solution: Let, S = Set of those families that read Science Today

R = Set of those families that read Readers Digest

C = Set of those families that read Caravan.

(i) You have to find $n(S' \cap R' \cap C')$

Let U = Set of the families questioned.

Now, $S' \cap R' \cap C' = (S \cup R \cup C)'$

$= U - (S \cup R \cup C)$

Therefore, $n(S' \cap R' \cap C') = n(U) - n(S \cup R \cup C)$

$= 100 - n(S \cup R \cup C)$

Now, $n(S \cup R \cup C) = 30 + 28 + 42 - 8 - 10 - 5 + 3 = 80.$

So, $n(S' \cap R' \cap C') = 100 - 80 = 20.$

(ii) You have to find $n[C - (R \cup S)]$

Now, $n[C - (R \cup S)] = n(C) - n[C \cap (R \cup S)]$

$= n(C) - n[(C \cap R) \cup (C \cap S)]$

$= n(C) - n(C \cap R) - n(C \cap S) + n(C \cap R \cap S)$

$= 42 - 10 - 5 + 3$

$= 30.$

(iii) You have to find $n[(S \cap C) - R]$

Now, $n[(S \cap C) - R] = n(S \cap C) - n(S \cap C \cap R)$

$= 5 - 3 = 2.$

Example 37: In a survey conducted of women it was found that:

(i) There are more single than married women in South Delhi.

(ii) There are more married women who own cars than unmarried women without them.

(iii) There are fewer single women who own cars and homes than married women without cars and without homes.

Is the number of single women who own cars and do not own homes greater than number of married women who do not own cars but own homes?

Solution: Let, A = Set of married women

B = Set of women who own cars

C = Set of women who own homes

Then, the given conditions are:

(i) $n(A') > n(A)$

(ii) $n(A \cap B) > (A' \cap B')$

(iii) $n(A \cap B' \cap C') > n(A' \cap B \cap C)$

We want to find $n(A' \cap B \cap C')$ and $n(A \cap B' \cap C)$

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Let U = Set of all women questioned.

Now, $A' = A' \cap U = A' \cap (B \cup B') = (A' \cap B) \cup (A' \cap B')$

$A = A \cap U = A \cap (B \cup B') = (A' \cap B) \cup (A \cap B')$

So, $n(A') = n(A' \cap B) + n(A' \cap B')$

$n(A) = n(A \cap B) + n(A \cap B')$.

By condition (i), we have:

$$n(A' \cap B) + (A' \cap B') > n(A \cap B) + n\,(A \cap B')$$

And by the condition (ii), we have:

$$n(A' \cap B) + n\,(A' \cap B') > n(A \cap B) + n(A \cap B') > n(A' \cap B') + n(A \cap B')$$

Therefore,

$$n(A' \cap B) > n\,(A \cap B').$$

Also, $A' \cap B = (A' \cap B) \cap (C \cap C')$

$= (A' \cap B \cap C) \cup (A' \cap B \cap C')$.

And, $A \cap B' = (A \cap B') \cap (C \cup C')$

$= (A \cap B' \cap C) \cup (A \cap B' \cap C')$

So, $n(A' \cap B) = n(A' \cap B \cap C) + n(A' \cap B \cap C')$.

$n(A \cap B') = n(A \cap B' \cap C) + n(A \cap B' \cap C')$.

This gives, using the condition (iii),

$$n(A \cap B \cap C) + n(A' \cap B \cap C') > n(A \cap B' \cap C) + n(A' \cap B \cap C)$$

i.e., $n(A' \cap B \cap C') > n(A \cap B' \cap C)$.

So, the number of single women who own cars and do not own a home is greater than the number of married women who do not own cars but own homes.

1.7 RELATION OF SETS

The concept of relation is studied between the elements of two sets. If A and B are two sets which are not empty(non-void) sets, then the relation **R** from set A to set B is represented as a **R** b, where a represents elements belonging to set A and b represents elements of set B.

It may be noted that the any relation from set A to set B is a subset of the Cartesian product of sets A and B.

It may be concluded from above that Relation from set A to set B can be defined as a set of ordered pairs (a, b) where a represents some elements belonging to set A and b elements of set B conforming to a rule.

Example 38

Consider set A and B as

A = {1, 2, 3}

B = {a,b,c}

Then one of the relations between Set A and B from A to B can be:

aRb = {(1,a),(1,c), (3,a), (3,c)}

It can be seen that it is a subset of AxB and it can also be depicted as given below:

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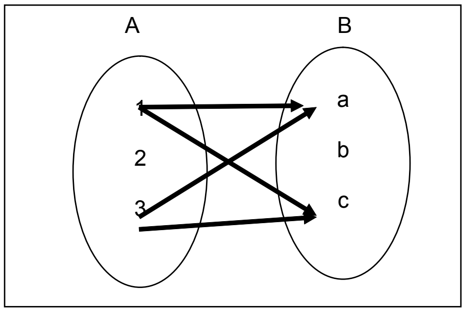

Let us consider a different example

Example 39: A = {1, 2, 3} and B = {1, 2, 3, 4, 5, 6, 7, 8, 9}.

Let us now state a relation between set A and B as those elements of B which are squares of elements of A; i.e. :

R = {(a,b): where b is square of a and a $\in$ A & b $\in$ B}. Then R is a set which has following elements

R = {(1,1), (2,4), (3,9)}

Therefore, it can be concluded that Relation is a linear operation which establishes relationship between the elements of two set's in accordance with a specific rule.

R : {(a, b) | (a, b) $\in$ A x B and a R b}

Similarly it can be discussed that a relation R(A, A) is a relation between set A with itself. In order to explain the concept, consider a set H comprising of all human beings as its elements. A set of sons and fathers would be a relation of H on H.

1.7.1 Number of Relations

Let A and B be two sets containing m and n number of elements respectively. Then AxB contains m x n elements. Since a relation of A on B is a subset of AxB, we can say that the number of subsets of AxB is

$$R_1^{mn} + R_2^{mn} + R_3^{mn} + \ldots\ldots\ldots\ldots + R_{mn}^{mn} = 2^{mn} - 1$$

Therefore, for A = {1, 2, 3) & B = {x, y}, the number of non-empty subsets or maximum number of relations is equal to $(2^6\text{-}1) = 63$.

1.7.2 Reflexive, Symmetric and Transitive Relations

There is a very important relation called Equivalence relation. Equivalence relation R on set A will have following three properties:-

(a) **Reflexivity**: A relation on a set A is reflexive if (x, x) $\in$ R for all x $\in$ A, i.e. xRx for every x in A.

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(b) **Symmetry**: A relation R is symmetric if whenever $(x, y) \in R$ then $(y, x) \in R$, i.e. xRy then yRx for every x & y belonging to A.

(c) **Transitivity**: A relation is transitive if whenever $(x, y) \in R$ and $(y, z) \in R$ then $(x, z) \in R$, i.e. if xRy and yRz then xRz.

Example 40: The relation "is equal to" is an equivalence relation on the set of real numbers since for any $x,y,z \in R$:

(a) $x = x$ (Reflexivity)

(b) If $x = y$, then $y = z$ (Symmetry)

(c) If $x = y$ and $y = z$, then $x = z$ (Transitivity).

Example 41: The relation "is greater than or equal to", is **Not** an equivalence relation on the set of real numbers since for any x,y,z R:-

(a) $x \geq x$ (Reflexivity)

(b) If $x \geq y$ then it is not necessary that $y \geq x$ (Symmetry not true).

(c) If $x \geq y$ and $y \geq z$, then $x \geq z$ (Transitivity is true).

Since e" relation is not symmetrical it is not an equivalence relation.

1.7.3 Equivalence Class

Commonly used symbol for denoting equivalence relation is ~. The three properties of equivalence relation can be writtens as given below using this notation:-

(a) $x \sim x$ for all x belonging to A.

(b) If $x \sim y$ then $y \sim x$.

(c) If $x \sim y$ and $y \sim z$ then $x \sim z$.

Now it can be stated that for an equivalence relation ~ on set A, we can define an Equivalence Class 'E' for an element 'a' belonging to A as:-

$E = \{b \mid b \sim a\}$.

That is a set E of all elements are b which are related to a by the relation ~.

It is to be noted that the equivalence class E for the element 'a' contains 'a' since $a \sim a$.

Theorem 1: Two equivalence classes E^1 and E^2 are either equal or disjoint.

Proof: Let E^1 be the equivalence class of element x^1, and let E^2 be the equivalence class determined by x^2. In order to prove that they are either disjoint or equal, let us start by saying that they are not disjoint. Let them have a common element y. Now, since y belongs to E^1 and E^2 both:-

$y \sim x^1$ and $y \sim x^2$.

By application of the property of symmetry we can conclude that

$x^1 \sim y$ and $x^2 \sim y$.

Application of property of transitivity we can conclude that

$x^1 \sim x^2$.

Now let y be an elemnet of E^1. It implies that $y \sim x^1$. By application of property of transitivity it can be stated that $y \sim x^2$. Therefore, it can be concluded that E^1 is a subset of E^2 and E^2 is a subset of E^1. Therefore

$E^1 = E^2$. Hence proved.

1.7.4 Partitions

A partition of set X is a collection of non-empty subsets of X which are disjoint and whose union together is complete X. That is collection of subsets X_1, X_2, X_3...... such that

(a) $X_i \neq \phi$ for i = 1, 2, 3 ....

(b) $X_i \cap X_j = \phi$ for all $i \neq j$

(c) $X_1 \cup X_2 \cup X_3 \cup ..... = X$

It can be noted that a partition, as the name suggests, divides a set X into pieces and each subset defined above is a partition of the set X.

Theorem 2: The equivalence classes of a relation form a partition of set X.

Proof: Consider an equivalence relation R on X, and we know that equivalence class cannot be empty and from above definition we know that if

$(x,y) \in R$ and then $[x]R \cap [y]R = \phi$.

Further for an element $x \in X$, it has been shown that $x \in [x]$ R. Therefore, union of all such equivalence classes covers X.

Theorem 3: A partition of a set X forms the equivalence relation (x,y) Î R if and only if there is an i such that x,y Î X_i.

Proof of the theorem is left for the reader to prove as an exercise.

(**Hint:** Show relation R is reflexive, symmetric, and transitive. The equivalence classes of this relation are the sets X_i).

Above two theorems combined together indicate that partition of a set and equivalence classes of the set are same. Therefore:-

(a) Every equivalence relation partitions its set into equivalence classes.

(b) Every partition creates an equivalence relation.

Check Your Progress

18. What do you understand by a domain and range?
19. Define a function. What are codomain and domain?
20. What do you understand by range of a function?
21. What are even and odd functions?

NOTES

1.8 FUNCTIONS

Functions are the special types of relations. Functions form a very basic concept of set theory because functions are utilised to depict a given problem and solving the function forms the solution of the problem.

A well defined gives a unique output for each input. Every function is defined for a set of inputs called its **Domain**. The domain of a function corresponds to set of possible outputs called the **Range.**

A function or mapping from set A to set B is a 'method' that pairs elements of set A with unique elements of set B and you denote $f : A \to B$ to indicate that f is a function from set A to set B.

B is called the *codomain* of the function f and A is called its *domain*. Also, for each element a of A, f defines an element b of B. Write it as $a \xrightarrow{f} f(a)$ or $a \xrightarrow{f} b$, $a \in A$, $b \in B$.

For example,

(*i*) The relation $f = \{(1, d), (2, c), (3, a)\}$ from $A = \{1, 2, 3\}$ to $B = \{a, c, d\}$ is a function from A to B. The domain of f is A and the codomain of f is B.

(*ii*) The relation $f = \{(a, b), (a, c), (b, d)\}$ from $A = \{a, b\}$ to $B = \{b, c, d\}$ is not a function.

Range of function: Let $f : A \to B$ be a function. The range of the function $R(f)$=$\{f\,(a : a \in A\}$. (Note that $R\,(f) \subseteq B$).

Notes:

1. From above example: $R(f)$ is $\{d, c, a\}$,
2. Let $f : R \to R^+$ be $f(x) = x^2$ (R^+, the set of positive real numbers). Clearly, f is a function whose domain is the set of real numbers and the co-domain is the set of positive real numbers.

$$R(f) \;=\; \{x^2 : x \in R\} = \{1, 4, 9,.....\}$$

Let $f : A \to B$ be a function f is said to be:

- ***One-to-one* (1–1) *function:*** If $x_1 \neq x_2$ then, $f(x_1) \neq f(x_2)$, $\forall\, x_1, x_2 \in A$.

  or

  Whenever $f(x_1) = f(x_2)$ then, $x_1 = x_2$. This function is also known as injective function.
- ***Onto surjective function:*** If for every element y in the codomain B, atleast one element x in the domain A such that $f\,(x) = Y$.

  or

  If $R(f)$ = Codomain B.
- ***Bijective function:*** If f is both 1–1 and onto function.

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- ***Constant function:*** If every element of the domain is mapped to a unique element of the codomain or the codomain consists of only one element.
- ***Into function:*** If atleast one element of the codomain is not mapped by any element of the domain.
- ***Identify function:*** If $f(x) = x$, $\forall\ x \in B$, in this case $A \leq B$. Sometimes, it is defined as $f : A \to A$ and $f(x) = x$, $\forall\ x \in A$.

For examples,

1. Let $f : R \to R$ be a function defined as $f(x) = 2(x + 2)$: Clearly, f is $1 - 1$ because if $2(x + 2) = 2\ (y + 2)$
$$\Rightarrow \quad 2x + 4 = 2y + 4$$
$$\Rightarrow \quad 2x = 2y \Rightarrow x = y$$
$$\therefore \quad f \text{ is } 1 - 1.$$
2. Define $f : R \to R^+$ by $f(x) = e^x$, $\forall\ x \in R$. Clearly, f is $1 - 1$ because if $f(x_1) = f(x_2)$
$$\Rightarrow \quad e^{x_1} = e^{x_2}$$
$$\Rightarrow e^{x_1 - x_2} = 1$$
$$\Rightarrow x_1 - x_2 = 0$$
$$\Rightarrow \quad x_1 = x_2$$
$$\therefore \quad f \text{ is } 1 - 1.$$
3. Let $A = \{5, 6, 7\}$ and $B = \{a, b\}$. Then the mapping $f : A \to B$ is defined as $f(5) = a$; $f(6) = b$; $f(7) = a$. Clearly, f is not $1 - 1$. But f is onto.
4. Consider the Example (2). If $f : R \to R^+$, defined by $f(x) = e^x$ is onto. Let x be any element IR^+, then $\log y \in IR$ such that $f(\log y) = e^{\log y} = y$.
5. Define $f : Z^+ \to Z^+$ as $f(n) = n^2$, $\forall\ n \in Z^+$. Clearly, f is an into mapping (not mapped by any element of Z^+) and $1 - 1$ mapping but f is not onto.
6. Define $f : Z \to Z$ by $f(n) = n + 1\ \ \forall\ n \in Z$. Clearly, f is $1 - 1$ and onto. For if, (*i*) $f(n) = f(m) \Rightarrow n + 1, = m + 1 \Rightarrow n = m \therefore f$ is $1 - 1$.
7. If n is any element of Z, then $n - 1 \in Z$ such that $f(n - 1) = n - 1 + 1 = n$. Hence f is onto.

Note: A one-one mapping of a set S onto itself is sometimes called a permutation of the set S.

Even and Odd Functions

If $f(x) = f(-x)$ f is called an even function and if

$f(x) = -f(-x)$ it is called an odd function.

Geometrically, the graph of an even fuction is symmetric with respect to the y-axis.

Inverse function: Let f be a bijective function from the set A to the set B. The inverse function of f is the function that is assigned to an element $b \in B$ the unique element a in A such that $f(a) = b$. The inverse function of f is denoted by f^{-1}. Hence $f^{-1}(b) = a$, when $f(a) = b$ (see Figure 1.6).

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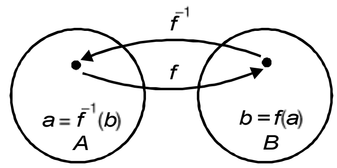

Fig. 1.6 *Inverse Function*

The function f^{-1} is the inverse function of f.

Note: A bijective function is called invertible since it can be defined as an inverse of this function.

Example 42:

(*i*) Define $f: Z \to Z$ by $f(n) = n + 1$. Is f invertible, and if it is what is its inverse?

Solution: The function f has an inverse, since it is a bijective function. Let y be the image of x, so that $y = x + 1$. Then $x = y - 1$, i.e., $y - 1$ is the unique element of Z that is sent to y by f. Hence $f^{-1} = y - 1$.

(*ii*) Let $A = \{a, b, c\}$, and $B = \{5, 6, 7\}$. Define $F: A \to B$ as $f(a) = 5$; $f(b) = 6$; $f(c) = 7$. Is f invertible, and if it is what is its inverse?

Solution: Clearly the given function is bijective. The inverse function f^{-1} of f is given as $f^{-1}(5) = a$; $f^{-1}(6) = b$; $f^{-1}(7) = c$.

(*iii*) Define $f: Z \to Z$ by $f(x) = x^2$. Is f invertible?

Solution: Since $f(-2) = f(2) = 4$, f is not 1–1. If an inverse function were defined, it would have to assign two elements to 2. Hence f is not invertible.

Compositions of functions: Let g be a function from the set A to the set B and let f be a function from the set B to the set C. The composition of the functions f and g denoted by $(f \circ g)$ is given in Figure 2.2 in such a way a that:

$$(f \circ g)\ (x) = f(g(x)) \quad \forall\ x \in A$$

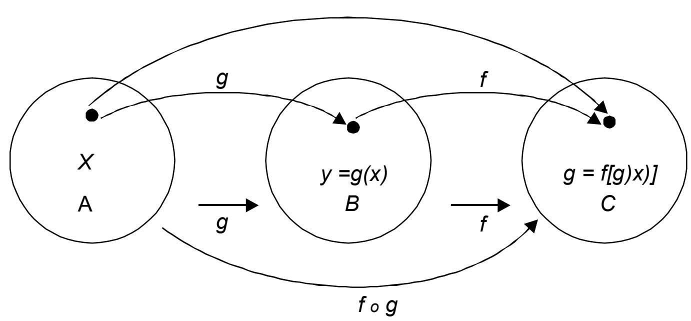

Fig. 1.7 *Composition of Function*

Example 43: Let $f: Z \to Z$ be a function defined by $f(x) = 2x + 3$. Let $g: Z \to Z$ be a function defined by $g(x) = 3x + 2$. Find (*i*) $f \circ g$ (*ii*) $g \circ f$.

Solution: Both $f\ o\ g$ and $g\ o\ f$ are defined. Further,

(*i*) $(f\ o\ g)\ (x) = f(g(x)) = f(3x + 2)$
$= 2(3x + 2) + 3 = 6x + 7$

(*ii*) $(g\ o\ f)\ (x) = g(f(x)) = g(2x + 3)$
$= 3(2x + 3) + 2 = 6x + 11$

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Eventhough $f\ o\ g$ and $g\ o\ f$ are defined, $f\ o\ g$ and $g\ o\ f$ need not be equal, i.e., the commutative law does not hold for the composition of functions.

Example 44: Let $A = \{1, 2, 3\}$, $B = \{x, y\}$, $C = \{a\}$. Let f: $A \to B$ be defined by $f(1) = x$; $f(2) = y$; $f(3) = x$. Let g: $B \to C$ be defined by $g(x) = a$; $g(y) = a$.

Find (*i*) $f\ o\ g$, if possible (*ii*) $g\ o\ f$, if possible.

Solution:

(*i*) $(f\ o\ g)\ (x) = f(g(x))$, but f cannot be applied on C and hence $f\ o\ g$ is meaningless.

(*ii*) $(g\ o\ f) : A \to C$ is meaningful. Now $(g\ o\ f)\ (x) = g(f(x))$, $\forall\ x \in A$.

$\therefore$ $(g\ o\ f)\ (1) = g(f(1)) = g(x) = a$
$(g\ o\ f)\ (2) = g(f(2)) = g(y) = a$
$(g\ o\ f)\ (3) = g(f(3)) = g(x) = a$

Result: If $f : A \to B$, g: $B \to C$ and h: $C \to D$, then $(h\ o\ g)\ o\ f = h\ o\ (g\ o\ f)$.

Proof: $[(h\ o\ g)\ o\ f]\ (x) = (h\ o\ g)\ (f(x))$
$= h[(g(f(x)]$ ...(1)

and $[h\ o\ (g\ o\ f)](x) = h[(g\ o\ f)\ (x)]$
$=h[g(f(x))]$ ...(2)

From Equations (1) and (2), $(h\ o\ g)\ o\ f = h\ o\ (g\ o\ f)$

Result: Let $f : A \to B$ and $g : B \to C$. Then,

(*i*) $g\ o\ f$ is onto, if both f and g are onto.

(*ii*) $g\ o\ f$ is $1 - 1$, if both f and g are $1 - 1$.

Proof:

(*i*) Let $z \in C$. Since $g : B \to C$ is onto, an element $y \in B$ such that $g(y) = z$. Since $f : A \to B$ is onto, for an element $x \in B$, such that $f(x) = y$.

$\therefore$ $(g\ o\ f)\ (x) = g(f(x)) = g(y) = z$.

$\therefore$ $(g\ o\ f)$ is onto.

(*ii*) Let $x_1 \neq x_2$ be two elements in A. Since $f : A \to B$ is one-one, and $f(x_1) \neq (x_2)$, $g(f(x_1)) = g(f(x_2))$. Thus, $g\ o\ f$ is one-one. In B, since g: $B \to C$ is one-one and $f(x_1) \neq (x_2)$.

1.9 FORMAL LOGIC

NOTES

Formal logic is a set of rules used in deductions which are self evident. Logic assumes something that can be True or False.

1.9.1 Mathematical Logic

One of the main aims of mathematical logic is to provide rules. The rules of logic give precise meaning to mathematical statements and distinguish between valid and invalid mathematical arguments. In addition, logic has numerous applications in computer science. These rules are used in the design of computer circuits, construction of computer programs, verification of the correctness of programs, and in many other ways.

Propositions: A proposition is a statement to which only one of the terms, true or false, can be meaningfully applied.
The value of a proposition if true is denoted by 1and if false is denoted by 0. Occasionally they are also denoted by the symbols T and F.

The following are propositions:

(*i*) $4 + 2 = 6$

(*ii*) 4 is an even integer and 5 is not.

(*iii*) 5 is a prime number.

(*iv*) New Delhi is the capital of India.

(*v*) $2 \in \{1, 3, 5, 7\}$

(*vi*) $42 \geq 51$

(*vii*) Paris is in England.

Of the above propositions, (*i*)–(*iv*) are true whereas (*v*)–(*vii*) are false.

The following are *not* propositions:

(*i*) Where are you going? (*ii*) $x + 2 = 5$

(*iii*) $x + y < z$ (*iv*) Beware of dogs

The expressions (*i*) and (*iv*) are not propositions since neither is true or false. The expression (*ii*) and (*iii*) are not propositions, since the variables in these expressions have not been assigned values and hence they are neither true or false.

Note: Letters are used to denote propositions just as letters are used to denote variables. The conventional letters used for this purpose are p, q, r, s, …

1.9.2 Logical Operators

There are several ways in which we commonly combine simple statements into compound ones. The words *or*, and, not, if… then and if and only if, can be added to one or more propositions to create a new proposition. New propositions are called compound propositions. Logical operators are used to form new propositions or compound propositions. These logical operators are also called connectives.

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Conjunction (AND): If p and q are propositions, then the proposition 'p and q', denoted by $p \wedge q$, is true when both p and q are true and is false otherwise. The proposition $p \wedge q$ is called the conjunction of p and q.

Example 45: Find the conjunction of the propositions p and q where p is the proposition 'Today is Sunday' and q is the proposition 'It is raining today'.

Solution: The conjunction of these two propositions is $p \wedge q,$ the proposition, 'Today is Sunday and it is raining today'.

Example 46: Let p be 'Ravi is rich' and let q be 'Ravi is happy'. Write each of the following in symbolic form:

(*i*) Ravi is poor but happy.

(*ii*) Ravi is neither rich nor happy.

(*iii*) Ravi is rich and unhappy.

Solution:

(*i*) $\sim p \wedge q$ (*ii*) $\sim p \wedge \sim q$ (*iii*) $p \wedge \sim q$

Disjunction (OR): If p and q are propositions, then disjunction p or q, denoted as $p \vee q,$ is false when p and q are both false and true otherwise. The proposition $p \vee q$ is called the disjunction of p and q.

Note that connectives $\sim$ and $\wedge$ defined earlier have the same meaning as the words 'not' and 'and' in general. However, the connective $\vee$ is not always the same as the word 'or' because of the fact that the word 'or' in English is commonly used both as an 'exclusive or' and as an 'inclusive or'. For example, consider the following statements:

(*i*) I shall watch the movie on TV or go to cinema.

(*ii*) There is something wrong with the fan or with the switch.

(*iii*) Ten or twenty people were killed in the fire today.

In statement (*i*), the connective 'or' is used in the exclusive sense; that is to say, one or the other possibility exists but not both. In (*ii*) the intended meaning is clearly one or the other or both. The connective 'or' used in (*ii*) is the 'inclusive or'. In (*iii*) the 'or' is used for indicating an approximate numbr of people, and it is not used as a connective. From the definition of disjunction it is clear that $\vee$ is 'inclusive or'.

Negation* (*NOT*)*: If p is a proposition, its negation not p is another proposition called the negation p. The negation of p is denoted by $\sim p$. The proposition $\sim p$ is read 'not p'.

Alternate symbols used in the literature are $\neg p$, $\overline{p}$ and 'not p'.

Note that a negation is called a connective although it only modifies a statement. In this sense, negation is the only operator that acts on a single proposition.

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Example 47: Find the negation of the propositions:

(*i*) It is cold.

(*ii*) Today is Sunday.

(*iii*) Ravi is poor.

Solution: The negation of the propositions are:

(*i*) It is not cold.

(*ii*) Today is not Sunday.

(*iii*) Ravi is not poor.

Conditional operator (if … then): Let p and q be propositions. The implication $p \to q$ is false when p is true and q is false and true otherwise. In the implication, p is called the *premise* or hypothesis and q is called the *consequence* or conclusion. Because implications arise in many places in mathematical argument, a wide variety of terminology is used to express $p \to q$. Some of the more common ways of expressing this implication are:

(*i*) p implies q. (*ii*) if p, then q. (*iii*) q if p. (*iv*) p only if q.

(*v*) p is sufficient for q. (*vi*) q whenever p. (*vii*) q is necessary for p.

We shall avoid the word 'implies' since it might be used in different contexts.

Example 48: Let p denote 'It is below freezing' and let q denote 'It is snowing'. Write the following statements in a symbolic form:

(*i*) If it is below freezing, it is also snowing.

(*ii*) It is not snowing if it is below freezing.

(*iii*) It is below freezing is a necessary condition for it to be snowing.

Solution: Recall that $p \to q$ can be read 'if p, then q' or 'p only if q' or 'q is necessary for p'. Then (*i*) $p \to q$ (*ii*) $p \to \sim q$ (*iii*) $q \to p$

Example 49: Let p and q be the propositions, where

p : You drive over 80 kms per hour.

q : You get a speeding ticket.

Write the following propositions in symbolic form:

(*i*) You will get a speeding ticket if you drive over 80 kms per hour.

(*ii*) If you do not drive over 80 kms per hour, then you will not get a speeding ticket.

(*iii*) Driving over 80 kms per hour is sufficient for getting a speeding ticket.

(*iv*) Whenever you get a speeding ticket, you are driving over 80 kms per hour.

Solution: (*i*) $p \to q$ (*ii*) $\sim p \to \sim q$ (*iii*) $p \to q$ (*iv*) $q \to p$

Biconditional operator (if and only if): Let p and q be propositions. The biconditional $p \leftrightarrow q$ is true when p and q have the same truth values and is false otherwise.

Note that the biconditional $p \leftrightarrow q$ is true when both the implications $p \rightarrow q$ and $q \rightarrow p$ are true. So 'p if and only if q' is used for biconditional. Other common ways of expressing the proposition $p \leftrightarrow q$ or $p = q$ are 'p is necessary and sufficient for q' and 'if p then q, and conversely'.

Each of the following theorems is well known, and each can be symbolized in the form $p \leftrightarrow q$:

(*i*) Two lines are parallel if and only if they have the same slope.

(*ii*) Two triangles are congruent if and only if all three sets of corresponding sides are congruent.

Example 50: Let p denote 'He is poor' and let q denote 'He is happy'. Write each of the following statement in symbolic form using p and q:

(*i*) To be poor is to be unhappy.

(*ii*) He is rich if and only if he is unhappy.

(*iii*) Being rich is a necessary and sufficient condition to being happy.

Solution:

(*i*) $p \leftrightarrow q$ (*ii*) $\sim p \leftrightarrow q$ (*iii*) $\sim p \leftrightarrow q$

1.9.3 Equivalence Formula

Two propositions are logically equivalent or simply equivalent if they have exactly the same truth values under all circumstances. We can also define this notion as follows:

The propositions p and q are called logically equivalent if $p \leftrightarrow q$ is a tautology. The equivalence of p and q is denoted by $p \Leftrightarrow q$.

Notes:

1. One way to determine whether two propositions are equivalent is to use a truth table. In particular, the propositions p and q are logically equivalent if and only if the columns giving their truth values agree.
2. Whenever we find logically equivalent statements, we can substitute one for another as we wish, since this action will not change the truth value of any statement.

NOTES

Table 1.2 Logical Equivalences

Equivalence	Name
1. $p \Leftrightarrow (p \vee p)$	Idempotents of $\vee$
2. $p \Leftrightarrow (p \wedge p)$	Idempotents of $\wedge$
3. $(p \wedge q) \Leftrightarrow (q \vee p)$	Commutativity of $\vee$
4. $(p \wedge q) \Leftrightarrow (q \wedge p)$	Communtativity of $\wedge$
5. $(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$	Associativity of $\vee$
6. $(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$	Associativity of $\wedge$
7. $\sim (p \vee q) \Leftrightarrow \sim p \wedge \sim q$	De Morgan's law 1
8. $\sim (p \wedge q) \Leftrightarrow \sim p \vee \sim q$	De Morgan's law 2
9. $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$	Distributive of $\wedge$ over $\vee$
10. $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$	distributive of $\vee$ over $\wedge$
11. $p \vee 1 \Leftrightarrow 1$	(Null) or Domination law 1
12. $p \wedge 0 \Leftrightarrow 0$	(Null) or Domination law 2
13. $p \wedge 1 \Leftrightarrow p$	Identity law 1
14. $p \vee 0 \Leftrightarrow p$	Identity law 2
15. $p \vee \sim p \Leftrightarrow 1$	Negation law 1
16. $p \wedge \sim p \Leftrightarrow 0$	Negation law 2
17. $\sim (\sim p) \Leftrightarrow p$	Double negation law (involution)
18. $p \rightarrow q \Leftrightarrow \sim p \vee q$	Implication law
19. $p \leftrightarrow q \Leftrightarrow (p \rightarrow q) \wedge (q \rightarrow p)$	Equivalence law
20. $(p \wedge q) \rightarrow r \Leftrightarrow p \rightarrow (q \rightarrow r)$	Exportation law
21. $(p \rightarrow q) \wedge (p \rightarrow \sim q) \Leftrightarrow \sim p$	Absurdity law
22. $p \rightarrow q \Leftrightarrow \sim q \rightarrow \sim p$	Contrapositive law
23. $p \vee (p \wedge q) \Leftrightarrow p$	Absorption law 1
24. $p \wedge (p \vee q) \Leftrightarrow p$	Absorption law 2
25. $p \leftrightarrow q \Leftrightarrow (p \wedge q) \vee (\sim p \wedge \sim q)$	Biconditional law

Example 51: Write an equivalent formula for $p \wedge (q \leftrightarrow r) \vee (r \leftrightarrow p)$ which does not contain the biconditional.

Solution: Since $p \leftrightarrow q \Leftrightarrow (p \rightarrow q) \wedge (q \rightarrow p)$, $p \wedge (q \leftrightarrow r) \vee (r \leftrightarrow p) \Leftrightarrow p \wedge ((q \rightarrow r) \wedge (r \rightarrow q)) \vee ((r \rightarrow p) \wedge (p \rightarrow r))$

Example 52: Write an equivalent formula for $p \wedge (q \leftrightarrow r)$ which contains neither the biconditional nor the conditional.

Solution: Since $p \leftrightarrow q \Leftrightarrow (p \rightarrow q) \wedge (q \rightarrow p)$ and $p \rightarrow q \Leftrightarrow \sim p \vee q$

$$p \wedge (q \leftrightarrow r) \Leftrightarrow p \wedge ((q \rightarrow r) \wedge (r \rightarrow q))$$
$$\Leftrightarrow p \wedge ((\sim q \vee r) \wedge (\sim r \vee q))$$

NOTES

Check Your Progress

22. What is a formal logic?
23. What are propositions?

1.10 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1. A set is a collection or compilation of distinct entities or objects. The objects which form the set are called the elements of the set.
2. The standard notation to list the elements of a set and denote the set is to use braces. For example set A having elements x, y & z is written as:

   A = {a,b,c}, similarly;

   B = {0, 1, 2, 3........} is set of whole numbers;

   N = { 1, 2, 3, 4, 5} is a set of first 5 natural numbers.

   X = { a, b, Y} is a set with elements a, b and the set Y. In this example Set Y is an element of set X.

   The above way of enumerating a set is called as Roster form.
3. A set that has no elements is called a null set or an empty set.
4. Two sets are said to be equal if they have the same elements. It may be noted that the order of elements in a set does not matter.

   If A = { a, b, c, d, 1, 2, 3} and
   B = { 1, a, 2, b, 3, d, c}, then A = B.
5. A set is finite, if it has a finite number of elements. The elements of such a set can be counted by a finite number. The number of elements in a finite set A is denoted by $n(A)$. Here n is a finite positive integer.
6. If a set has an infinite number of elements it is an infinite set. The elements of such a set cannot be counted by a finite number.
7. Set A is called a subset of B if all elements of A are also the elements of B. It is denoted as given below:

   $A \subseteq B$

   In other words it can be said that A = B if and only if $A \subseteq B$ and $B \subseteq A$.
8. A set Y is a proper subset of a set X if and only if $Y \subset X$ and $Y \neq X$ and $Y \neq \phi$.

   Every set is a subset of itself. Thus, $X \subseteq X$.
9. If X, Y, Z are sets and $X \subseteq Y$ and $Y \subseteq Z$, then $X \subseteq Z$.

NOTES

10. Cardinality of a set A is the number of elements of set A and is denoted as |A|.
11. Union of two sets X & Y is said to be the set of all elements of set X along with all elements of set Y. It is denoted as 'U'. X U Y is a set containing all elements that belong to set X, or to set Y or to both.

    X U Y = { x| x $\in$ X or x $\in$Y}
12. Difference of two sets X & Y is the set of all elements that belong to X but do not belong to Y.

    $$X - Y = \{ x | x \in X, x \notin Y\}$$
13. Complement of a set X in the Universal set U is defined as a set of all elements that belong to U but do not belong to X.
14. Two sets are said to be disjoint, if $AÇB = \{\}$. As in (iii) above.
15. The identity under set union is the empty set F and the identity under intersection is the universal set U.

    $X \cup \Phi = \Phi \quad U \quad X = X$

    $Y \cap U = U \cap Y = Y$ (note U denotes universal set)
16. DeMorgan's laws state that:

    (a) $(X \cup Y)^c = X^c \cap Y^c$. (b) $(X \cap Y)^c = X^c \cup Y^c$.
17. For any set X,

    $X \cup X = X$ and $X \cap X = X$
18. A well defined gives a unique output for each input. Every function is defined for a set of inputs called its Domain. The domain of a function corresponds to set of possible outputs called the Range.
19. A function or mapping from set A to set B is a 'method' that pairs elements of set A with unique elements of set B and you denote $f : A ® B$ to indicate that f is a function from set A to set B.

    B is called the *codomain* of the function f and A is called its *domain*.
20. Range of function: Let $f : A \to B$ be a function. The range of the function $R(f)$={$f(a : a \in A\}$. (Note that $R(f) \subseteq B$).
21. If $f(x) = f(-x)$ f is called an even function and if

    $f(x) = -f(-x)$ it is called an odd function.
22. Formal logic is a set of rules used in deductions which are self evident. Logic assumes something that can be True or False.
23. A proposition is a statement to which only one of the terms, true or false, can be meaningfully applied.

    The value of a proposition if true is denoted by 1and if false is denoted by 0. Occasionally they are also denoted by the symbols T and F.

1.11 SUMMARY

NOTES

- A set is a collection or compilation of distinct entities or objects. The objects which form the set are called the elements of the set.
- Two sets are said to be equal if they have same elements.
- An empty set Ô is a subset of all sets. As also each set is subset of itself.
- Cardinality of a set A is the number of elements of set A and is denoted as |A|.
- Complement of a set X in the Universal set U is defined as a set of all elements that belong to U but do not belong to X.
- Difference of two sets X & Y is the set of all elements that belong to X but do not belong to Y.
- Union and intersection of sets are distributive and associative.
- The identity under set union is the empty set Ô and the identity under intersection is the universal set U.
- The union operation is distributive over intersection and intersection operation is distributive over union.
- Intersection, union and difference correspond to logical operations AND, OR and NOT, respectively.
- The roster or tabulation method describes a set by listing each element of the set within the braces.
- A set is well defined if and only if it can be decided that a given object is an element of the set.
- A diagrammatic representation of set operation is called Venn diagram.
- Relation is a linear operation which establishes relationship between the elements of two set's in accordance with a specific rule.
- A relation *R* on a set *A* is called an equivalence relation, if *R* is reflexive, symmetric and transitive.
- A relation *R* on a set *A* is said to be a partial order relation, if *R* is reflexive, anti-symmetric and transitive.
- Equivalence relation C on set A is reflexive, symmetric and transitive.
- Two equivalence classes E^1 and E^2 are either equal or disjoint.
- A partition of set X is a collection of non-empty subsets of X which are disjoint and whose union together is complete X
- Every equivalence relation partitions its set into equivalence classes.
- Every partition creates an equivalence relation.

NOTES

- Every function is defined for a set of inputs called its Domain. The domain of a function corresponds to set of possible outputs called the Range.
- In mathematics, a function is a relation between a set of elements called the domain and another set of elements called the codomain.
- The mathematical function describes the dependence between two quantities. One of the quantity being known and is termed as the independent variable, argument of the function, or its 'input' while the second quantity is produced and is termed as the dependent variable, value of the function, or 'output'.
- The function can be defined in various ways using a formula, by a plot or graph, by an algorithm that computes it, or by a description of its properties.
- A proposition is a statement to which only one of the terms, true or false, can be meaningfully applied.
- Logical operators are used to form new propositions or compound propositions.

1.12 KEYWORDS

- **Set:** A set is a collection of any type of objects, things or numbers.
- **Elements:** The constituent members of a set are called elements of the set.
- **Roser or tabulation method:** It is the method to describe a set in which each element of the set is enlisted within the braces.
- **Finite set:** It is the type of set in which there is a finite number of elements.
- **Infinite set:** It is the type of set in which there is an infinite number of elements.
- **Null set:** It is the set which contains no elements.
- **Disjoint sets**: If two sets have no elements in common, then they are disjoint sets.
- **Universal set:** It is the set of all objects relevant to a particular application.
- **Equivalent sets:** They refer to sets in which there is a one-to-one correspondence between the elements.
- **Domain:** In mathematics, the domain (sometimes called the source) of a given function is the set of 'input' values for which the function is defined.
- **Codomain:** in mathematics, the codomain or target set of a function is the set Y into which all of the output of the function is constrained to fall. It is the set Y in the notation $f : X ® Y$. The codomain is also sometimes referred to as the range but that term is ambiguous as it may also refer to the image.
- **Function:** Function is a way of expressing relation between two sets by use of symbols representing variables and symbols which define operations on variables.

- **Mathematical function:** The mathematical function describes the dependence between two quantities.

1.13 SELF ASSESSMENT QUESTIONS AND EXERCISES

Short Answer Questions

1. Define set in minimum words.
2. What is a subset?
3. What is universal set?
4. Mention three basic operations on sets.
5. A set has three elements. How many subsets can it form?
6. What is the resultant set of two sets *A* and *B* when these are disjoint?
7. Define the term set equivalent.
8. Draw a Venn diagram of a three intersecting nonempty sets *A, B* and *C* having a universal set *U*.
9. Differentiate between minset and maxset.
10. Define the concept of relation with reference to set theory.
11. What is equivalence class?
12. Define the term function.
13. What are into and onto functions?
14. Define one-to-one function.
15. What is another name for a function which is both one-one and onto?
16. Define inverse function.
17. Define the term mathematical logic.
18. What are logical operators? Define the various logical operators.
19. Define equivalence formula.

Long Answer Questions

1. Check the distributive laws for $\cup$ and $\cap$ and DeMorgan's laws.
2. Determine if the following statements are true or false. Where A represents any set.

   (*i*) $\varphi \subseteq A$

   (*ii*) $A \subseteq A$

   (*iii*) $(A) = A$

   (*vi*) $\{1, 3, 4\}$ is a subset of $\{3, 2, 1, 4\}$.

NOTES

(*v*) {3, 2, 1, 4} is a subset of {3, 1, 2,}.

(*vi*) The empty set is a subset of every set.

(*vii*) 1 is an element of {3, 2, 1, 4}.

(*viii*) A - (A - B) = B.

(*xi*) A ∩ (B - C) = (A ∩ B) - (A ∩ C).

(*x*) A ⊆ B and A ⊆ C then it implies that A ⊆ (B ∩ C).

(*xi*) (A ∩ B) U (A - B)= A.

(*xii*) If A ⊆ B and A ⊆ C then A ⊆ (B U C).

(*xiii*) A - (B - A) = A - B.

(*xiv*) A ⊆ B or A ⊆ C then it implies that A ⊆ (B ∩ C).

(*xv*) A U (B - C) = (A U B) - (A U C).

(*xvi*) If A ⊆ B or A ⊆ C it implies that A ⊆ (B U C).

3. Prove that two points (x_0, y_0) and (x_1, y_1) on a plane are equivalent if $y_0 - x_0^2 = y_1 - x_1^2$. Further describe the equivalence classes.
4. Give an example of a set whose elements are sets.
5. Let A be a non-empty set. $P(A)$ denotes the set of all subsets of A. $P(A)$ is called as power set of A. Find the power set of the following sets.

   (*i*) $A = \{a, b\}$ (*ii*) $B = \{1, 2, 3\}$
6. Suppose the given universal set $\mu = \{1, 2, ..., 10\}$

   Express the following sets with bit strings where ith bit in the string is 1 if i is in the set and 0 otherwise.

   (*i*) $A = \{4, 5, 6\}$ (*ii*) $B = \{1, 2, 9, 10)$ (*iii*) $C = \{2, 3, 7, 8, 9\}$

   Use bit strings to find:

   (*i*) $A \cup B$ (*ii*) $A \cup C$ (*iii*) $B \cup C$ (*iv*) $A \cap B$
   (*v*) $A \cap C$ (*vi*) $B \cap C$ (*vii*) $A \cup B \cup C$ (*viii*) $A \cap B \cap C$
7. Let $\{A_1, A_2, ..., A_n\}$ be a partition of a set X and let B be a nonempty subset of X.

   Prove that $\{A_i \cap B \,/\, A_i \cap B = \phi\}$ is a partition of $X \cap B$.
8. How many different relations are there from a set of n elements to a set of m elements?
9. Let R and S be two relations which are reflexive, symmetric and transitive. Show that $R \cap S$ is also reflexive, symmetric and transitive.
10. How inverse functions are defined in case of trigonometric functions?
11. If a function f: $[1, \infty) \rightarrow [1, \infty)$ is defined as $f(x) = 2^{x(x-1)}$, then what is $f^{-1}(x)$?

12. Find whether each of the following functions from $\{1, 2, 3, 4\}$ to itself is 1–1.
    (*i*) $f(1) = 2; f(2) = 1; f(3) = 3; f(4) = 4$
    (*ii*) $f(1) = 2; f(2) = 2; f(3) = 4; f(4) = 3$
    (*iii*) $f(1) = 4; f(2) = 2; f(3) = 3; f(4) = 4$.
13. Show that p is equivalent to the following formulae:
    $\sim p \rightarrow p \wedge p \rightarrow p \vee p \rightarrow p \vee (p \wedge p)\ p \wedge (p \vee q)\ (p \wedge q) \vee (p \wedge \sim q)$
    $(p \vee q) \wedge (p \vee \sim q)$
14. Which of the following propositions generated by p, q and r are equivalent to one another?
    (*i*) $(p \wedge r) \vee q$ (*ii*) $p \vee (r \vee q)$ (*iii*) $r \wedge p$ (*iv*) $\sim r \vee q$
    (*v*) $(p \vee q) \wedge (r \vee q)$ (*vi*) $r \rightarrow p$ (*vii*) $r \vee \sim p$ (*viii*) $p \rightarrow r$

NOTES

1.14 FURTHER READINGS

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NOTES

UNIT 2 SET THEORY AND NUMBER SYSTEM

Structure

2.0 INTRODUCTION

For any study of topology it is essential that basic logical and mathematical concepts be studied and understood. In the previous unit we have discussed elementary concepts of set theory towards study of logical concepts. In this unit we will discuss the basic mathematical concepts required for our study of Topology, i.e., the Number System.

You will study axioms of real numbers, Absolute value or modulus and Interval notation. You will understand positive and negative real numbers discussing set and section of integers, well ordering property, Archimedean ordering property and strong induction principle. You will describe Cartesian product of sets mentioning Cartesian square and power, infinite products, Cartesian product of functions and two sets. You will analyse various types of sets, such as finite, infinite, countable and uncountable sets.

2.1 OBJECTIVES

NOTES

After going through this unit, you will be able to:

- Construct concept of number system with naïve and axiomatic approach
- Differentiate in real numbers, integers, natural numbers, rational and irrational numbers
- Interpret axioms and properties of real numbers, modulus and interval notation
- Understand well ordering property, Archimedean ordering property and strong induction principle
- Explain Cartesian product of a family of sets and infinite products
- Describe concept of finite, infinite, countable and uncountable sets

2.2 CONSTRUCTION OF NUMBER SYSTEM

There are two ways of understanding and building the concepts of Number system. The Naive approach and the Axiomatic approach.

2.2.1 The Naive Approach

In this Naive Set Theory approach is to build the Real number system from the basics of set theory. In this approach each type of number system is assumed to be a set. Properties of the set are assumed and the reasons for such assumptions are stated. However, this approach is more based on logical understanding and less on mathematical concern.

2.2.2 The Axiomatic Approach

A second way of approaching the study of number system is to assume that the number systems exist with certain unambiguously stated properties. Then it is essential to prove these properties. In other words it is required that the number system be described by some axioms and then prove these axioms. In this chapter we will follow this approach. We will state few axioms for the real number system and then prove the properties.

2.3 SET OF REAL NUMBERS

Rationals and irrationals together constitute the real number system.

Any real number is either rational or irrational. Any point on the number line is related to some corresponding real number uniquely and each real number is related to some corresponding point on the number line uniquely.

The classification of the set of all real numbers is shown in Figure below.

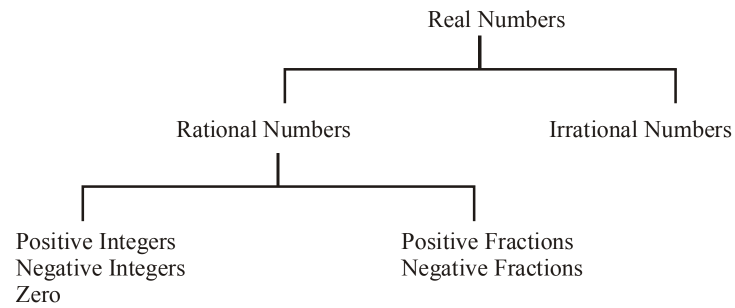

Fig. 2.1 *Classification of Real Numbers*

The real number system can be presented as a deductive logical system. By the use of the properties of this system and the operations performed therein we can understand the reasoning behind any steps in mathematical methods connected with real numbers. The operations on real numbers are applicable to rational and irrational numbers.

2.3.1 Axioms of Real Numbers

The real number system is fundamental to mathematical analysis. It may be defined as a complete, simply ordered field thus suggesting that it is a system which is (*i*) a field (*ii*) simply ordered, and (*iii*) complete.

We divide the axioms for real numbers into three groups.

1. The axioms of the field.
2. The axioms of order.
3. The axioms of completeness.

The meanings of fields, simple order and completeness will be clear from the axioms which are discussed below and which are used in all the familiar algebraic manipulations. In practice these laws are used with great rapidity so that often we are not even aware of the fact that we are using them.

1. **The Axioms of the Field:** A field is a set of numbers which contains the sum, the difference, the product and the quotient (except division by zero) of any two numbers in the set. When subjected to these operations the resulting numbers continue to remain in the field.

   There are six axioms of a field for addition and multiplication.

   Let a, b, c be real numbers. Then we have the following axioms.

(*i*) ***The Axiom of Closure:*** There is a *unique* real number $a + b$ which is the result of addition of b to a. Similarly there is a unique real number $a\,.b$ which is obtained by multiplying a by b.

The sum $a + b$ is a unique real number

The product $a\,.b$ is a unique real number.

(*ii*) ***The Axiom of Commutation:*** $a + b$ and $b + a$ are the same real number. Similarly $a.b$ and $b.a$ are the same real number.

$$a + b = b + a,\ a + (b + c) = (b + c) + a,\ -6 + 11 = +11 - 6$$

$$a.b = b.a, \qquad a(bc) = (bc)a, \qquad 3.5 = 5.3$$

(*iii*) ***The Axiom of Association:*** Any two of a, *b, c,* may be added to the third and the result will always be the same, viz., $a + b + c$ which is uniquely defined.

The product of any two of a, *b, c,* by the third is the same viz., *abc* which is uniquely defined.

$$a + (b + c) = (a + b) + c = (a + c) + b = a + b + c$$

$$a\,(bc) = (ab)\,c = (ac)\,b = abc$$

$$5 + (3 + 4) = 3 + 4 + 5,\ 5(3.4) = 3.4.5$$

(*iv*) ***The Axiom of Distribution:*** The distribution law connects the two operations of addition and multiplication.

The product of a and $b + c$ is the same real number as the sum of ab and ac.

$$a(b + c) = ab + ac;\ (a + b)\,(c + d) = ac + ad + bc + bd$$

(*v*) ***The Axiom of Identity:*** The addition of zero to any number does not change the number. Zero is called the identity element for addition as it leaves the number unchanged.

$$a + 0 = 0 + a = a$$

The multiplication by 1 leaves the number unchanged. 1 is the identity element for multiplication. For each and every a, a . 1 and a are the same real number.

$$a\,.\,1 = 1\,.\,a = a$$

(*vi*) ***The Inverse Axiom:*** For all a,

$$a + (-a) = 0$$

$-a$ is called the additive inverse of $+a$

$+a$ is called the additive inverse of $-a$

For all $a,\ a \neq 0$

$$a \cdot \frac{1}{a} = 1$$

$\frac{1}{a}$ is called the multiplicative inverse of a

a is called the multiplicative inverse of $\frac{1}{a}$.

The following statements may be regarded as theorems.

1. If $a + a = a$ then a is equal to 0

Proof: $\qquad a + a = a$ $\qquad$ Hypothesis

$\therefore \qquad a + a + (-a) = a + (-a)$

$\therefore \quad a + [a + (-a)] = a + (-a)$ Association

$\therefore \quad a + 0 = 0$

$\therefore \quad a = 0$

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2. If a is a real number then $a.0 = 0$

   Proof: $0 + 0 = 0$

   $\therefore \quad a(0 + 0) = a.0$

   $\therefore \quad a.0 + a.0 = a.0$ Distribution

   $\therefore \quad a.0 = 0$

3. If $a = b$, then $-a = -b$

4. There is no real number a which is a reciprocal of 0.

   Proof: Let there be a real number a which is a reciprocal of 0, then $a.0 = 1$.
   But $a.0 = 0$ so that $0 = 1$ which is absurd (Every real number a, other than zero has a unique reciprocal $1/a$).

5. If $a + c = b + c$ then $a = b$

   Proof: $a + c + (-c) = b + c + (-c)$

   $\therefore \quad a + [c + (-c)] = b + [c + (-c)]$ Association

   $\therefore \quad a + 0 = b + 0$

   $\therefore \quad a = b.$

6. If $ac = bc$ and $c \neq b$ then $a = b$

   Proof: $(ac)\dfrac{1}{c} = (bc)\dfrac{1}{c}$

   $\therefore \quad a\left(c \cdot \dfrac{1}{c}\right) = b\left(c \cdot \dfrac{1}{c}\right)$ Association

   $\therefore \quad a.1 = b.1$

   $\therefore \quad a = b$

7. If $a + b = 0$, then $b = -a$

   Proof: $-a + (a + b) = -a + 0$

   $\therefore \quad (-a + a) + b = -a + 0$

   $\therefore \quad 0 + b = -a + 0$

   $\therefore \quad b = -a.$

8. If $ab = 1$, $a \neq 0$, then $b = \dfrac{1}{a}$

Proof: $(ab)\ \dfrac{1}{a} = 1\ .\ \dfrac{1}{a}$

NOTES

$\therefore \quad b\left(a \cdot \dfrac{1}{a}\right) = 1 \cdot \dfrac{1}{a}$

$\therefore \quad b \cdot 1 = 1 \cdot \dfrac{1}{a}$

$\therefore \quad b = \dfrac{1}{a}$

9. $b(-a) = -(ab)$.
10. $(-a)(-b) = ab$.
11. $a(b - c) = ab - ac$

    Proof: $a(b - c) \quad = a[b + (-c)]$

    $= a\ .\ b + a(-c)]$

    $= ab - ac.$
12. $a - 0 = a$.
13. $a + a = 2a$.
14. $\dfrac{a}{b} \cdot \dfrac{c}{d} = \dfrac{ac}{bd}$ if $b \neq 0, d \neq 0$.
15. $-(a + b) = -a - b$.
16. $b + (a - b) = a$.
17. $-\dfrac{a}{b} = -\dfrac{-a}{b} = \dfrac{a}{-b}$.
18. $\dfrac{a+b}{c} = \dfrac{a}{c} + \dfrac{b}{c}$.
19. $-a + (-a) = (-2)\ .\ a$
20. If $b = ax$, then $x = \dfrac{b}{a},\ a \neq 0$
21. $(ab)^{-1} = a^{-1}\, b^{-1}$, i.e., the reciprocal of the product is the product of the reciprocals.

    Proof: $(ab)(a^{-1}\, b^{-1}) = a[b\,(a^{-1}\, b^{-1})] = a[bb^{-1})a^{-1}]$

    $= a\,[1\ .\ a^{-1}] = a\ .\ a^{-1} = 1$. Hence, the result.
22. If $\dfrac{a}{b} = \dfrac{c}{b}$, then $a = c,\ b \neq 0$

23. $-(a + b) = (-a) + (-b)$
24. $(x - a)(x + a) = x^2 - a^2$
25. $(a - b)^2 = a^2 + 2ab + b^2$

NOTES

2. The Axioms of Order: A field is said to be *ordered* if it satisfies the properties of trichotomy, transitivity, addition and multiplication. This means that real numbers can be thought of as being arranged on the number line.

A real number a is positive if $a > 0$ and negative if $a < 0$.

Zero is neither positive nor negative.

If a, b are real then $a < b$ if and only if $b > a$.

There are six axioms of an ordered field.

Let a, b, c be real numbers.

(*i*) ***The Axiom of Trichotomy***

If a, b are real then one and only one of the following statements is true: $a < b$, $a = b$, $a > b$

(*ii*) ***The Axiom of Transitivity***

If $a > b$ and $b > c$ then $a > c$

If $a < b$ and $b < c$ then $a < c$

(*iii*) ***The Axiom of Addition***

If $a > b$ then $a + c > b + c$

(*iv*) ***The Axiom of Multiplication***

If $a > b$ and $c > 0$ then $ac > bc$

(*v*) ***The Axiom of Extent:*** If $a < b$ there exist real numbers c, d such that

$c < a, b < d$

(*vi*) ***The Axiom of Density:*** If $a < b$ there exists a real number c such that

$a < c < b$

Thus, between any two real numbers, however close to each other, there is always a real number.

26. $a > b$ if and only if $-a < -b$

Proof: $a > b \quad \therefore \quad a + [(-a) + -(b)] > b + [(-a) + (-b)]$

$\therefore \quad [a + (-a)] + (-b) > (b + (-b)] + (-a)$

$\therefore \quad 0 + (-b) > 0 + (-a)$ or $-b > -a$.

27. If $a > 0$, then $-a < 0$.
28. If $a > 0$ and $b > 0$, then $ab > 0$
29. If $a > b$ and $c >$ d, then $a + c > b + d$.

NOTES

Proof: $a + c > b + c, \quad b + c > b + d.$

$\therefore \quad a + c > b + d.$

30. If $a > b$ and $c < 0$, then $ac < bc$

Proof: $a(-c) > b(-c)$

$\therefore \quad -ac > -bc$

$\therefore \quad ac < bc.$

An inequality is reversed if we multiply both sides by the same negative number.

Thus $-5 > -7$ gives $7 > 5$ or $5 < 7$

Absolute Value or Modulus: Some quantities like price, demand, supply never take negative values while other quantities like profit, utility, etc., may be either positive or negative. Often we need to use quantity regardless of its sign. It is called the absolute value of the quantity.

The absolute value of a real number a is denoted by $|a|$. Thus $|-2| = 2$, $|2| = 2$. The absolute value of a real number a is defined by the following conditions.

$$|a| = a \quad \text{if} \quad a > 0$$
$$|a| = -a \quad \text{if} \quad a < 0$$
$$|a| = 0 \quad \text{if} \quad a = 0$$

The square of the absolute value of a number is the square of the number.

$$|a|^2 = |a^2|, |-a|^2 = a^2$$

31. For every a, $|-a| \leq a \leq |a|$
32. $|ab| = |a|\,|b|$ Thus $|5.3| = |5|\,|3| = 15$.
33. $|a + b| \leq |a| + |b|$. Thus $|5 + 3| \leq |5| + |3| = 8$.

    and $\quad |5 + (-3)| < |5| + |-3|$ is < 8.
34. $|a| \geq a$ for every a.
35. $|-a| = |a|$ for every a.
36. Let $a > 0$ then $|x| < a$ if and only if $-a < x < a$.

Proof: If $x > 0$ then $-a < x$, i.e., $|x| < a$ when $-a < x < a$

If $x < 0$ then $-a < x$, i.e., if $|x| < a$ then $-a < x < 0$ (refer Figure below)

Combining we get $-a < x < a$

$|x| < a$ implies $-a < x < a$

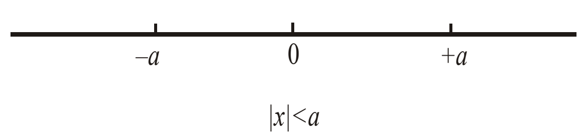

Fig. 2.2 $|x|<a$

NOTES

3. **The Axiom of Completeness:** This axiom is a subtle characterising property of the real number system. It is sometimes called the least upper bound axiom. It states that every set which has an upper bound has a least upper bound. Every set which has a lower bound has a greatest lower bound.

Consider the approximate values of $\sqrt{3}$ .There are several approximations ; 1.7, 1.73, 1.732,....

Since $1.7 < 1.73 < 1.732 < ... < \sqrt{3}$, i.e., each value is less than $\sqrt{3}$ and none equals $\sqrt{3}$. We say $\sqrt{3}$ is an upper bound of the set.

$$\{1.7, 1.73, 1.732,...\}$$

There are other upper bounds but $\sqrt{3}$ is the best. It is the least upper bound.

An upper bound *t* of a set is said to be the least upper bound if no upper bound is less than *t*.

A lower bound *v* is the greatest lower bound if no lower bound is greater than *v*.

The set of positive integers has no upper bound.

The set of negative integers has no lower bound.

Interval Notation: An interval is a special type of point set. Since there are an infinite number of real numbers between any two real numbers, intervals of real numbers are also infinite sets.

A *closed interval* is written $[a, b]$ or $a \leq x \leq b$ where $a < b$. The end points *a, b* of a closed interval are elements of the interval, i.e., *a*, *b* belong to the interval. Thus *x* can take the values *a, b* and all the values between *a*, *b*.

An *open interval* is written (a, b) or $a < x < b$ where $a < b$. The end points *a*, *b* do not belong to the interval. Here *x* can take any values between the numbers *a*, *b*.

A partially open, i.e., half open, half closed interval is written $a < x \leq b$ or $(a, b]$.

A partially open, i.e., half closed; half open interval is written $a \leq x < b$ or $[a. b)$.

An interval $a \leq x \leq b$ where $a = b$ is called a degenerate interval. It reduces to a single point on a line.

An interval $a \leq x \leq b$ where $a > b$ is called a null interval. The null interval contains no points. For example, $4 \leq x \leq 0$ is null since both the inequalities $4 \leq x$ and $x \leq 0$ cannot be satisfied for any number *x*.

An interval may be bounded or unbounded. If an interval contains all points less than a certain number *b*, it is said to be unbounded from below. It is written $-\infty < x \leq b$ in the closed form and written $\infty < x < b$ in the open form.

If an interval contains all points more than a certain number a, it is said to be unbounded from above. It is written $a \leq x < \infty$ in the closed form and written $a < x < \infty$ in the open form.

If a, b are given numbers, an interval $a \leq x \leq b$ is said to be bounded.

If an interval extends to infinity on either side it is said to be unbounded and written $-\infty < x < \infty$.

2.3.2 Summary of Properties of Algebraic Operations on Real Numbers

Let a, b, c be real numbers.

Properties of +

(*i*) The sum $a + b$ is unique	*Closure and Uniqueness*
(*ii*) $a + b = b + a$	*Commutation*
(*iii*) $(a + b) + c = a + (b + c)$	*Association*
(*iv*) $a + 0 = a$ for all a	*Identity*
(*v*) $a + (-a) = 0$ for all a	*Inverse*

Properties of ×

(*i*) The product $a \cdot b$ is unique	*Closure and Uniqueness*
(*ii*) $a \cdot b = b \cdot a$	*Commutation*
(*iii*) $(a \cdot b)\, c = a\, (b \cdot c)$	*Association*
(*iv*) $a \cdot 1 = a$ for all a	*Identity*
(*v*) $a \cdot \dfrac{1}{a} = 1$ for all $a\ (\neq 0)$	*Inverse*

Properties of =

(*i*) $a = a$	*Reflexivity*
(*ii*) $a = b$ implies $b = a$	*Symmetry*
(*iii*) $a = b$ and $b = c$ implies $a = c$	*Transitivity*

Properties of <

(*i*) One and only one of the following is true: $a = b$, $a < b$, $b < a$	*Trichotomy*
(*ii*) If $a < b$ and $b < c$ then $a < c$	*Transitivity*
(*iii*) If $a < b$ then $a + c < b + c$	*Addition*
(*iv*) If $a < b$ and $c > 0$ then $ac < bc$	*Multiplication*
(*v*) For real a, b however close there exists a real c such that $a < c < b$	*Density*

Similarly for *properties* of >, the same will be the properties but with > symbol.

NOTES

Check Your Progress

1. What is the real number system?
2. What are the various groups of the axioms of the real number system?
3. What is a field in set theory?
4. State the axiom of order.
5. What do you understand by absolute value of the quantity?
6. What is the axiom of completeness?

2.4 POSITIVE AND NEGATIVE REAL NUMBERS

Let us define number $p \in R$ if $p > 0$ and to be negative if $p < 0$. The set of all positive real Numbers is denoted as R_+ and the set of all real non-negative real numbers as $\check{R}_+$.

2.4.1 Set of Integers Z

Before we define the set of integers, let us define an inductive set. A set X of real numbers is said to be inductive if it satisfies following two properties:-

(a) Number $1 \in X$.

(b) If $x \in X$, then $x + 1$ also $\in X$

Let us assume a set X comprising of all inductive subsets of R, then Z_+; a set of all positive integers; is collection of all elements of X. This can be proved because R_+ is inductive, since it has 1 as element and the property that if $x > 0$ then $x + 1 > 0$ means that $x + 1$ is an element of R_+ for all x.

Therefore Z_+ is a subset of R_+.

It implies that the elements of Z_+ are all positive and the least number is 1. Therefore, it is an inductive set. From this we now define Z the set of integers as a set whose elements are 0 (zero), all elements of Z_+ and negatives of all elements of set Z_+. Following can be proved for the set of integers Z:-

(a) Addition, subtraction and multiplication of integers would result in an integer.

(b) Division of integers may or may not result in an integer.

(c) For any integer x, there is no integer y which is $x < y < x+1$.

2.4.2 Section of Integers

Let n be a positive integer and let S_n denote the set of all positive integers less than n, then S_n is called a section of positive integers. Therefore, elements of S_n are all integers from 1 to N-1.

NOTES

2.4.3 Well Ordering Property

Theorem 1: It states that every non-empty subset of Z_+ has a smallest element.

Proof: Let X be a set which is subset of Z_+ and has no least number. It implies that integer 1 $\notin$ X, since otherwise it would have been the least integer of X. Now, let Y be a set of positive integers which is = X^c. Then 1 must belong to Y. Let us assume n a positive integer such that all integers $\leq$ n $\in$ Y. It can be stated that n + 1 must belong to X since it is complement of Y. However, then n+1 should be the least element of X, since all positive integers less than N+1 are elements of Y. This is contradictory to our first assumption, thus it can be concluded that X is an empty set.

2.4.4 Strong Induction Principle

Theorem 2: If X is a set of positive integers and if $S_n \subset Z_+$ means that n $\in$ X, , then X = Z_+.

Proof : Let us state that X $\neq Z_+$. Let m be the smallest positive integer that is not in X. So m $\notin$ X. Then every positive integer less than m is in X. From definition of section of positive integers it implies that $S_m \subset$ X. Then in accordance with the hypothesis m $\in$ X which is contrary to assumption. Therefore It implies that X = Z_+.

2.4.5 Archimedean Ordering Property

It states that the set Z_+ does not have an upper bound in the set of Real Numbers R.

Proof: Let us assume that Z_+ has an upper bound. Since $Z_+ \subset$ R, from property 7 it implies that it has a least upper bound, let it be 'u'. Then there is a positive integer n such that it is greater than u – 1, otherwise u would not be upper bound. Then n + 1 > u which cannot be true. Therefore Z_+ does not have an upper bound.

Check Your Progress

7. Define an inductive set.
8. What is meant by section of integers?
9. State well ordering property.
10. What does Archimedean ordering property state?

2.5 CARTESIAN PRODUCT OF SETS

Let *A* and *B* be two sets. The Cartesian product of *A* and *B* is defined as:

$$A \times B = \{a, b)/a \in A;\ b \in B\}$$

i.e., the set of all ordered pairs (a_i, b_j) for every $a_i \in A$; $b_j \in B$.

NOTES

For example, $A = \{1, 2\}, B = \{a, b, c\}$

$$A \times B = \{(1, a), (2, a), (1, b), (2, b), (1, c), (2, c)\}$$

$$B \times A = \{(a, 1), (a, 2), (b, 1), (b, 2), (c,1), (c, 2)\}$$

(Clearly $A \times B \neq B \times A$)

Note: You can represent the Cartesian product as a rectangular array having n rows and m columns labelled in order as $a_1, a_2, ..., a_n$ and $b_1, b_2,...,b_n$ respectively.

In the field of mathematics, product of two sets is known as Cartesian product. It is in fact, a direct product for these two sets. The name of such a product bears the name of René Descartes who formulated analytic geometry. Cartesian product of two sets A, which is a set of points on x-axis, and B, which is set of all points on y-axis, is given as $A \times B$. It shows a set of ordered pairs having first component that belongs to set A and second component belonging to set B and it represents the entire $x - y$ plane. A Cartesian product is mathematically shown as:

$$A \times B = \{(x, y) \mid x \in A \text{ and } y \in B\}$$

A pack playing cards contains 52 elements having four sets as club, hearts, diamond and spade, with each containing 13 elements, 2, 3, 4, 5, 6, 7, 8, 9, 10, Jack, Queen, King and Ace. Thus, there are two sets, one set having four suits and each suit is a set of thirteen elements, and the corresponding Cartesian product contains $13 \times 4 = 52$ elements of all possible playing cards {(2, Spade), (3, Spade), ..., (Ace, Spade), (2, Hearts), ..., (King, Club), (Ace, Club)}.

A table can represent Cartesian product of two finite sets where one is the set of rows whereas another is the set of columns. This forms ordered pairs that refer to cells in this table. An element of set is located in a cell by mentioning row and column number. Cartesian products of two sets are not commutative as you find in cases product of two numbers. This can happen only when two input sets are the same.

Let there be four sets M, N, P and Q.

Cartesian product is commutative when sets of two inputs are the same. If different, they are not commutative since ordered pairs are reversed and even if same elements are there, pairing differs.

This is mathematically expressed as:

$$M \times N \neq N \times M$$

For example,

$\{2, 5\} \times \{6, 7\} = \{(2, 6), (2, 7), (5, 6), (5, 7)\}$ and $\{6, 7\} \times \{2, 5\} = \{(6, 2), (6, 5), (7, 2), (7, 5)\}$

But one exception is there, if you take an empty set or equal sets.

$$M \times \phi \neq \phi \times M$$

Also, if there are two equal sets M and N and $M = N$, then

$$M \times N = N \times M$$

Cartesian product is not associative: This fact is expressed mathematically as follows:

$$(M \times N) \times P \neq M \times (N \times P)$$

Cartesian product with intersections: Here we can interchange B and C and intersection with product as shown follows:

$$(M \cap N) \times (P \cap Q) = (M \times P) \cap (N \times Q)$$

But this may not be true, if you use union instead of intersection.

$$(M \cup N) \times (P \cup Q) \neq (M \times P) \cup (N \times Q)$$

Cartesian product is distributive over intersection and union: This fact is expressed as follows:

$$(M) \times (N \cap P) = (M \times N) \cap (M \times P) \text{ and,}$$
$$(M) \times (N \cup P) = (M \times N) \cup (M \times P)$$

n-ary Product

Generalization can be made for a Cartesian product over n sets $A_1, ..., A_n$:

This is mathematically expressed as:

$$A_1 \times .... \times A_n = \{(a_1, ..., a_n) : a_i \in A_i\}$$

It shows an n-tuples set. If these tuples are in form of ordered pairs in a nested structure, then these are given as $(A_1 \times ... \times A_{n-1}) \times A_n$.

Cartesian Square and Cartesian Power

Cartesian square of a set P is a binary Cartesian product $P^2 = P \times P$. For example, if R represents a set of real numbers, two-dimensional plane can be shown as $R^2 = R \times R$ and all points or any point as an ordered pair of (x, y) where $x \in R$ and $y \in R$. This, in fact, denotes a coordinate system in two dimensions.

Cartesian power of a set P is defined as:

$$P^n = P \times P \times P \times .... \times P = \{(x_1, x_2, ... x_n) \mid x_1 \in P \wedge x_2 \in P \wedge x_3 \in P \wedge .... \wedge x_n \in P\}$$

..... n times

If you take the example of a set of real numbers R, then R^n is the general case and $R^3 = R \times R \times R$ is the case of a three-dimentional plane.

Infinite Products

You can define a Cartesian product for any number of sets when n becomes very large, may be tending to infinity. If I denotes a set of index, then collection of index set is given as $\{X_i \mid i \in I\}$ indexed by I, then Cartesian product will be:

$$\prod_{i \in I} X_i \{f : I \to \bigcup_{i \in I} X_i \mid (\forall i)(f(i) \in X_i)\},$$

Here, the set containing all functions has its definition on index set as a function's value corresponding to particular index i may be taken as an element that belongs to X_i.

Also, in case of each j contained in I, you may show the projection as:

$$\pi j : \prod_{i \in I} X_i \to X_j,$$

And is defined as $\pi_j(f) = f(j)$.

This is known as *jth projection map.*

You can show an important case when the index set is the set N of natural numbers. In such a case, Cartesian product has a set containing infinite sequences having ith term that corresponds to set X_i. Consider elements of,

$$\prod_{n=1}^{\infty} \mathbb{R} = \mathbb{R}^{\omega} = \mathbb{R} \times \mathbb{R} \times \ldots$$

This may be viewed as vector having infinite number of components in the set of real numbers.

There is occurrence of Cartesian exponentiation of special case when all X_i factors involved in the product are the same set X. In this case,

$$\prod_{i \in I} X_i = \prod_{i \in I} X$$

Denotes the set of all functions mapped from I to X. Such a situation is important for studying cardinal exponentiation.

Seeing in the light of facts as presented for infinite Cartesian product, finite Cartesian products are taken to be a special case. In such an interpretation, an n-tuple is function on $\{1, 2, 3, 4, ..., n\}$ assuming value at i as ith element of the tuple.

2.5.1 Cartesian Product of Functions

NOTES

If function f is defined as $f : X \rightarrow Y$ and g as $g : A \rightarrow B$, their Cartesian product is defined as $f \times g : (X \times A) \rightarrow (Y \times B)$ as,

$$(f \times g)\ (a,\ b) = (f\ (a),\ g(b)$$

Such a definition may also get extended to tuples as well as collections of functions tending to infinity.

2.5.2 Cartesian Product of Two Sets

Let A and B be two sets. The set of all ordered pairs (a, b) such that $a \in A$, $b \in B$ is called the Cartesian product of A and B and is denoted by $A \times B$.

Notes: 1. The ordered pair (a, b) is not the same as the set $\{a, b\}$.

2. Two ordered pairs (a, b) and (c, d) are equal if and only if $a = c$ and $b = d$.
3. For the ordered pair (a, b), a is called the first coordinate and b is second coordinate.

For example, let $A = \{1, 2, 3\}$, $B = \{4, 5\}$. Then $A \times B = \{(1, 4), (1, 5), (2, 4), (2, 5), (3, 4), (3, 5)\}$ and $B \times A = \{(4, 1), (5, 1), (4, 2), (5, 2), (4, 3), (5, 3)\}$. This shows that $A \times B \neq B \times A$.

Example 1: Let A be any set. Then $A \times \phi$ and $\phi \times A$ are empty sets.

Solution: Suppose if possible $A \times \phi$ be not empty.

Then, there is some $x \in A \times \phi$

By definition, $\qquad x = (a, b) \quad$ where $a \in A$, $b \in \phi$.

This is absurd as ϕ is empty.

Hence, $A \times \phi = \phi$. Similarly $\phi \times A = \phi$.

Example 2: Let A, B, C be three sets. Then,

$$A \times (B \cup C) = (A \times B) \cup (A \times C)$$

Solution: Let $x \in A \times (B \cup C)$.

Then, $x = (a, d)$ where $a \in A$, $d \in B \cup C$

Now, if $d \in B$, then $x \in A \times B$

and if $d \in C$, then $x \in A \times C$

In any case, $x \in (A \times B) \cup (A \times C)$

Therefore, $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$

Similarly, $(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$

This proves the result.

Example 3: Let A, B, C be three sets. Then,

$$A \times (B - C) = (A \times B) - (A \times C)$$

Solution: Let $(a, d) \in A \times (B - C)$

Then, $a \in A$, $d \in B$ and $d \notin C$

So, $(a, d) \in A \times B$ and $(a, d) \notin A \times C$

Therefore, $(a, d) \in (A \times B) - (A \times C)$

This proves $A \times (B - C) \subseteq (A \times B) - (A \times C)$.

To prove the other side containment we proceed as under:

Let, $x \in (A \times B) - (A \times C)$

Then, x is an element of $A \times B$ but it does not belong to $A \times C$.

This means that $x = (a, b)$ where $a \in A$, $b \in B$ but $b \notin C$,

Otherwise, $x = (a, b) \in A \times C$

This implies that, $b \in B - C$.

Therefore, $x \in A \times (B - C)$

Hence, $(A \times B) - (A \times C) \subseteq A \times (B - C)$.

Example 4: If the set A has m elements and the set B has n elements, how many elements does $A \times B$ have?

Solution: Let $a \in A$. Then number of elements of $A \times B$ with first coordinate as a is n. But a can be chosen in m ways. So, the number of distinct elements of $A \times B$ is mn.

Example 5: If $A = \{1, 4\}$, $B = \{4, 5\}$, $C = \{5, 7\}$

Find (*i*) $(A \times B) \cup (A \times C)$ (*ii*) $(A \times B) \cap (A \times C)$

Solution: (*i*) By Example 1.36, $(A \times B) \cup (A \times C) = A \times (B \cup C)$

Now, $B \cup C = \{4, 5, 7\}$

So, $$A \times (B \cup C) = \{(1, 4), (4, 4), (1, 5), (4, 5), (1, 7), (4, 7)\} = (A \times B) \cup (A \times C)$$

(*ii*) Now, $A \times B = \{(1, 4), (4, 4), (1, 5), (4, 5)\}$,
$A \times C = \{(1, 5), (1, 7), (4, 5), (4, 7)\}$

So, $(A \times B) \cap (A \times C) = \{(1, 5), (4, 5)\}$.

2.6 FINITE AND INFINITE SETS

Infinite and finite sets are intuitively understood as sets having infinite or finite elements. However, certain aspects of these sets need to be discussed to a greater length.

From previous section it can be remembered that if n is a positive integer then section S_n is a set of all positive integers less than n. Intuitively S_n can be understood as a finite set. A set X is called a finite set if there exists a bijective correspondence of all elements of X with a subset of Z_+.

In other words, X is finite if,

(a) It is an empty set. Or

(b) There is a bijection function $f : X \rightarrow\rightarrow \{1, 2, ..., n\}$.

It can be noticed that in case of (a) above cardinality of X = 0 and in case of (b) cardinality of x is n.

Lemma 1: Consider a set of positive integers X. Let x_i be an element of set X. Then there is a bijective function f of the set X to the set (1, 2, ...m } where m is a positive integer; if and only if there is a bijective function f_1 of the set X - $\{x_i\}$ with the set $\{1, 2, ...m\text{-}1\}$.

Proof: In order to prove the above postulate let us begin by first assuming that a bijective function f_1 exists such that,

$f_1 : X - \{x_i\} \rightarrow\rightarrow \{1, 2, ...m\text{-}1\}$.

Let there be a function $f: X \rightarrow\rightarrow \{1, 2, ...m\}$

For all $f(a) = f_1(a)$ for all $a \in X - \{x_i\}$ and

$f(x_i) = m$.

It can straight away be noticed that f is bijective. In order to prove the converse, let there exist a bijective function f such that

$f: X \rightarrow\rightarrow \{1, 2, ...m\}$

Now, if f maps x_i to number integer m, then the condition $f: X - \{x_i\}$ is the desired bijective function of $f - \{x_i\}$ with $\{1, 2, ... m\text{-}1\}$.

Let $f(x_i) = n$, and x^1 be the point of X such that $f(x^1) = m$. Then, $x_i \neq x^1$.

Define a new function $g : X \rightarrow\rightarrow \{1, 2, ...m\}$ such that

$g(x_i) = m$

$g(x^1) = n$

$g(x) = f(x)$ for all $x \in X - \{x_i\} - \{x^1\}$

From above it can be noted that g is a bijection function and the restriction $g : X - \{x_i\}$ is the bijection of $X - \{x_i\}$ with $\{1, 2, ...m\text{-}1\}$.

2.7 COUNTABLE AND UNCOUNTABLE SETS

Intuitively it can be said that sets whose cardinality can be stated are countable and the others are uncountable. However, there is a concept of countably infinite sets that needs to be discussed here.

A set is said to be countably infinite if its elements can be put in one-to-one correspondence with the set of positive integers Z_+. For example a set of integers given as {...-3, -2, -1, 0, 1, 2, 3, ...} is infinite but since we can count all elements it is called as countably infinite set.

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Therefore, it can be stated that a set X is said to be countably infinite if there exists a bijection function such that

$$f : X \rightarrow\rightarrow Z_+.$$

Taking the above example for the set of integers it can be seen that there exists a bijection function f: $Z \rightarrow\rightarrow Z_+$ and it is defined as:

$f(n) = 2n$ for all $n > 0$ and

$f(n) = -2n + 1$ for all $n \leq 0$.

Thus a set X is said to be countable if it is either finite or countably infinite. If the set is not countable then it is called uncounlable.

Theorem 3: Let X be a non-empty set. Then the following are equivalent.

(a) X is countable.

(b) There is a surjective function $f : Z_+ \rightarrow\rightarrow X$.

(c) There is an injective function $f_1 : X \rightarrow\rightarrow Z_+$.

Proof: **(a) → (b)** If X is countably infinite, then there exists a bijection f : $Z_+ \rightarrow\rightarrow X$. Therefore (b) is proven. If X is finite, then there is bijection function g such that :

$g : \{1, \ldots, m\} \rightarrow X$ for some m.

Then the function $f : Z_+ \rightarrow\rightarrow X$ can be exteneded to be given as:

$f(i) = g(i)$ for $1 \leq i \leq n$, and

$f(i) = g(n)$ for $i > n$.

(b) → (c) Let $f : Z_+ \rightarrow\rightarrow X$ be a surjection, then we state that there exists an injective function h: $X \rightarrow\rightarrow Z_+$. We define

$h(x) =$ smallest element of $f^{1}(\{x\})$

Since f is surjective $f^{1}(\{x\})$ is a non-empty set. Therefore, h is well defined. Now if $x \neq x^1$ then the sets $f^{1}(\{x\}) \cap f^{1}(\{x^1\}) = \phi$, which implies that

$\min^{-1}(\{x\}) \neq \min^{-1}(\{x^1\})$. Hence

$f_1(x) \neq f_1(x^1)$ and $f_1 : X \rightarrow\rightarrow Z_+$ is injective.

(c) → (a) Assume that $f_1 : X \rightarrow\rightarrow Z_+$ is an injection. We want to probe that X is countable. By changing the range of f_1, we have $f_1: X \rightarrow f_1(X)$ which is a bijection function of X. Now if we can prove that every subset of Z^+ is countable we can prove our theorem.

Lemma 2: If X is an infinite subset of Z_+ then X is countably infinite.

Proof: Let X be an infinite subset of Z_+. Then we define a function

$g : Z_+ \rightarrow\rightarrow X$ as follows. Let $g(1) =$ smallest element of X. Since X is infinite, it is a non-empty set. Therefore, function g: is well defined.

Now assuming g(1) .... g(m-1) are well defined we have

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$g(m)$ = smallest element of $\{X : \{g(1), \ldots, g(m-1)\}$. Now since X is an infinite set and $\{X : \{g(1), \ldots, g(m-1)\}$ is a non-empty set, g(m) is well-defined.

Now, $g(m + k) > g(m)$ for all $m, k \in Z_+$.

By definition $g(m + 1) > g(m)$ for all $m \in Z_+$.

Then setting $A = \{k \in Z_+: g(m + k) > g(m)\}$ we note

$1 \in A$ and if $g(m+ (k - 1)) > g(m)$, then $g(m + k) > g(m + (k - 1)) > g(m)$.

Therefore, $A = Z_+$. Implies $g(m + k) > g(m)$ for all $m, k \in Z_+$.

Now assuming $n,m \in Z_+$ and that $n > m$ such that $n = m + k$. By the above $g(n) = g(m + k) > g(m)$ proving that g is an injection function. Next we prove that g is a surjection. To do this we first show that $g(m) \geq m$.

Let $A = \{m \in Z_{+:}\ g(m) \geq m\}$. It can be seen that, $1 \in A$. Now If $x \in A$, then

$g(x + 1) > g(x) \geq m$ so that $g(x + 1) \geq x + 1$. Hence $x + 1 \in A$. Now by the principle of mathematical induction $A = Z_+$.

Now take $x^0 \in X$. We have to show that $g(y^0) = x$ for some $y^0 \in Z_+$. If $x^0 = 1$, then $y^0 = 1$. So assume that $x^0 \geq 2$. Consider the set $B = \{x \in X | g(x) \geq x^0\}$. Since $g(x^0) \geq x^0$, the set B is non-empty and by well-ordering principle set B has a smallest element.

Let y^0 = the smallest element D. If $y^0 = 1$, then $g(y^0)$ = smallest element of $X \leq x^0 \leq g(y^0)$. Thus $g(y^0) = x^0$. It may be safe to assume that x> smallest element of X. Then $g(y^0) \geq x^0 > g(y^0 - 1) > \ldots > g(1)$. Therefore $g(y^0) = x^0$ implies that the function g is a surjection.

Lemma 3: If two sets X and Y are countably infinite, then $X \times Y$ countably infinite.

Proof: X and Y are countably infinite, therefore there exists a surjective functions $f : Z_+ \rightarrow \rightarrow X$ and $f_1 : Z_+ \rightarrow \rightarrow Y$. Let $g : Z_+ \times Z_+ \rightarrow \rightarrow X \times Y$ by

$F(x,y) = (f(x), f_1(y))$. The function F is surjective. Since $Z_+ \times Z_+$ is countably infinite, there is a bijection $h : Z_+ \rightarrow \rightarrow Z_+ \times Z_+$. Then $G : Z_+ \times X \times Y$ defined by $G = F \bullet h$ is a surjection. Therefore the set $X \times Y$ is countable.

Lemma 4: A subset of a countable set is countable.

Proof: Let X be a countable set and let Y be a subset of X.

Then there exists an injective function f of X on Z_+. Now the restriction of the injective function f onto Y is an injection of Y on Z_+.

Lemma 5: The set $Z_+ \times Z_+$ is countable.

Proof: By the theorem proved above it would be adequate to define an injective function $f : Z_+ \times Z_+ \rightarrow \rightarrow Z_+$.

Let $f : Z_+ \times Z_+ \rightarrow \rightarrow Z_+ = f(x,y) = 2^x 3^y$.

Let $2^x 3^y = 2^k 3^l$. If $x < k$, then $3^y = 2^{k''x} 3^l$. The left hand side of this equation is an odd number whereas the right hand side is an even number which is not possible. Therefore $x = k$ and $3^y = 3^l$. Then also $y = l$. Hence f is injective.

NOTES

Check Your Progress

11. Give definition of the Cartesian products.
12. Give mathematical representation for the Cartesian product.
13. What is Cartesian square?
14. What is countably infinite set

2.8 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1. The real number system is fundamental to mathematical analysis. It may be defined as a complete, simply ordered field thus suggesting that it is a system which is (*i*) a field (*ii*) simply ordered, and (*iii*) complete.
2. We divide the axioms for real numbers into three groups.
   - The axioms of the field.
   - The axioms of order.
   - The axioms of completeness.
3. A field is a set of numbers which contains the sum, the difference, the product and the quotient (except division by zero) of any two numbers in the set.
4. A field is said to be *ordered* if it satisfies the properties of trichotomy, transitivity, addition and multiplication. This means that real numbers can be thought of as being arranged on the number line.
5. Quantity regardless of its sign is called the absolute value of the quantity.

   The absolute value of a real number a is denoted by $|a|$. Thus $|-2| = 2$, $|2| = 2$.
6. It is sometimes called the least upper bound axiom. It states that every set which has an upper bound has a least upper bound. Every set which has a lower bound has a greatest lower bound.
7. A set X of real numbers is said to be inductive if it satisfies following two properties:-

   (a) Number $1 \in X$. (b) If $x \in X$, then $x + 1$ also $\in X$
8. Let n be a positive integer and let S_n denote the set of all positive integers less than n, then S_n is called a section of positive integers.
9. It states that every non-empty subset of Z_+ has a smallest element.
10. It states that the set Z_+ does not have an upper bound in the set of Real Numbers R.
11. Let A and B be two sets. The Cartesian product of A and B is defined as:
    $A \times B = \{a, b)/a \in A;\ b \in B\}$
    i.e., the set of all ordered pairs (a_i, b_j) for every $a_i \in A$; $b_j \in B$.

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12. A Cartesian product is mathematically shown as:
$A \times B = \{(x, y) \mid x \in A \text{ and } y \in B\}$
13. Cartesian square of a set P is a binary Cartesian product $P^2 = P \times P$. For example, if R represents a set of real numbers, two-dimensional plane can be shown as $R^2 = R \times R$ and all points or any point as an ordered pair of (x, y) where $x \in R$ and $y \in R$. This, in fact, denotes a coordinate system in two dimensions.
14. A set is said to be countably infinite if its elements can be put in one-to-one correspondence with the set of positive integers Z_+. For example a set of integers given as {...-3, -2, -1, 0, 1, 2, 3, ...} is infinite but since we can count all elements it is called as countably infinite set.

2.9 SUMMARY

- The real number system may be defined as a complete, simply ordered field.
- A field is a set of numbers which contains the sum, the difference, the product and the quotient (except division by zero) of any two numbers in the set.
- A field is said to be *ordered* if it satisfies the properties of trichotomy, transitivity, addition and multiplication.
- Let A and B be two sets. The Cartesian product of A and B is defined as:
  $A \times B \quad = \quad \{a, b)/a \in A; b \in B\}$
  i.e., the set of all ordered pairs (ai, bj) for every ai $\in$ A; bj $\in$ B.
- A Cartesian product is mathematically shown as:
  $A \times B = \{(x, y) \mid x \in A \text{ and } y \in B\}$
- A binary operation on a set R is a function that maps Cartesian product or R with R onto R.
- Addition, subtraction and multiplication of integers would result in an integer. Division of integers may or may not result in an integer.
- Let n be a positive integer and let S_n denote the set of all positive integers less than n, then S_n is called a section of positive integers.
- Every non-empty subset of Z_+ has a smallest element. This property is called well ordering property.
- Archimedean Ordering Property states that the set Z_+ does not have an upper bound in the set of Real Numbers.
- A set is said to be countably infinite if its elements can be put in one-to-one correspondence with the set of positive integers Z_+.
- Thus a set X is said to be countable if it is either finite or countably infinite. If the set is not countable then it is called uncounlable.
- If X is an infinite subset of Z_+ then X is countably infinite

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2.10 KEY WORDS

- **Real numbers:** Rational numbers and irrational numbers together constitute the real numbers.
- **Ordered field:** A field is said to be ordered if it satisfies the properties of trichotomy, transitivity, addition and multiplication.
- **Absolute value or modulus:** A quantity regardless of its sign is called the absolute value of the quantity.
- **Interval notation:** An interval is a special type of point set.
- **Completeness or supremum property:** It states that every set which has an upper bound has a least upper bound. Every set which has a lower bound has a greatest lower bound.
- **Linear continuum:** It is a linearly ordered set of more than one element that is compactly ordered.
- **Countable infinity:** A set is said to be countably infinite if its elements can be put in one-to-one correspondence with the set of positive integers.
- **Uncountable set:** A set is said to be uncountable if it contains too many elements to be countable.
- **Infinite set:** If a set has an infinite number of elements it is an infinite set.

2.11 SELF ASSESSMENT QUESTIONS AND EXERCISES

Short Answer Questions

1. Write a short note on the axioms of the field with its proofs.
2. Describe the axioms of order briefly.
3. Briefly describe absolute value of the quantity.
4. Discuss the axiom of completeness in short.
5. Describe Cartesian square briefly.
6. Write a short note on infinite products.
7. Give a brief account of finite and infinite sets.

Long Answer Questions

1. Discuss properties of real numbers with axiomatic approach.
2. Give a general account of positive and negative real numbers.
3. Explain Cartesian product of sets and their properties.
4. Give a general account of countable and uncountable sets.

5. Prove the following for real numbers R, using properties defined in this unit:
   (*i*) If $a + b = a$, then $a = 0$.
   (*ii*) $a(b + c) = ab + ac$.
   (*iii*) $(-1)\, a = - a$.
   (*iv*) $a\,(b - c) = ab - ac$.
   (*v*) If $a \neq 0$ and $ab = a$, then $b = 1$.
   (*vi*) $(a/b)(c/d) = ac/bd$ for b & $d \neq 0$.
   (*vii*) If $a > 0$ and $b > 0$, then $a + b > 0$ and $ab > 0$.
   (*viii*) If $a > b$ and $c < 0$, then $ac < bc$.
6. Prove that for a set comprising of inductive sets, the intersection its elements is inductive.
7. Show that every nonempty subset of Z that is bounded above has a largest element.
8. Show there is a bijective correspondence of $X \times Y$ with $Y \times X$.
9. Show that finite cartesian products and finite unions of finite sets are finite.
10. Depict the injective mapping of
    f: $\{a, b, c\} \rightarrow\rightarrow \{a, b, c, d, e\}$. Check if they are bijective.
11. Show that if X is not finite and $X \subset Y$, then Y is not finite.
- If X and Y are finite, check if the set of all functions f: $X \rightarrow\rightarrow Y$ is finite.
- Show that a countable union of countable sets is countable.

2.12 FURTHER READINGS

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UNIT 3 INFINITE SETS

NOTES

Structure

3.0 INTRODUCTION

The concept of finite and infinite sets was discussed in the previous unit. We know that an infinite set is a set whose elements cannot be counted. It can also be said that a set which does not have an upper bound is infinite. More formally, we have studied that a set can be called infinite if it has a bijection with a proper subset of itself or if it has countably infinite subsets. In this unit you will study about infinite sets and infinite set theorem. You will discuss well-ordered sets with their properties and examples and well-ordering theorem. The maximum principle, maximal element and Zorn's lemma are also explained in this unit.

3.1 OBJECTIVES

After going through this unit, you will be able to:

- Comprehend concept of infinite and finite sets
- Understand the importance of axiom of choice and its equivalence
- To state and prove the existence of choice function
- Understand the concept of totally ordered and partially ordered sets
- State and prove the well ordering theorem
- State and prove the maximum principle
- Comprehend the equivalence of Zorn's lemma and axiom of choice

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3.2 INFINITE SETS

If a set has an infinite number of elements it is an infinite set. The elements of such a set cannot be counted by a finite number. A set of points along a line or in a plane is called a point set. A finite set has a finite subset. An infinite set may have an infinite subset.

Examples of infinite sets: A few examples of infinite sets are given below:

(a) A set X comprising of all points in a plane is an infinite set.

(b) Z = {……… -2, -1, 0, 1, 2, ……….} i.e. set of all integers is an infinite set.

(c) Set of all positive integers which are multiple of 5 is an infinite set.

Check Your Progress

1. Define an infinite set.
2. Give some examples of infinite sets.

3.3 INFINITE SET THEOREM

Theorem 1: Let X be a set such that:

(a) There exists an injective function $f : Z_+ \to\to X$.

(b) There exists a bijection of X with a proper subset of itself.

(c) X is infinite.

Proof: In order to prove the theorem we will first prove that if (a) is true it implies (b), then we will prove that if (b) is true it implies (c), then finally if (c) is true it implies (a).

Let f be an injective function defined as

$f : Z_+ \to\to X$. Let the image set $f(Z_+)$ be denoted by Y; and let f(n) be denoted by x_n. Now since f is injective, for all $n \neq m$, $x_n \neq x_m$.

Now let us assume a function f^1

$$f^1 : X \to\to X - \{x_1\} \text{ such that}$$
$$f^1(x_n) = x_{n+1} \text{ for all } x_n \in Y$$
$$f^1(a) = a \text{ for all } a \in X - Y$$

It can be seen from the above that the function f^1 is a bijection.

Now for the second step; let us assume that Y is a proper subset of X and

$f: X \to\to Y$ is a bijection. By assumption, there is a bijection $h: X \to\to \{1, 2, ...n\}$ for some n. Then the composite $h \circ f^{-1}$ is a bijection of Y with {1, 2, ...n}. This is a contradiction and cannot be true.

For the third step, we assume that X is infinite and define an injective function $f : Z_+ \rightarrow\rightarrow X$ by principle of induction. The set X is a non-empty set, so let us assume an element x_1 of X such that it is equal to f(1).

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The set $X - f(\{1, 2, ...n-1\})$ is a non-empty set because if it was empty, then the function $f : \{1, 2, ... n-1\} \rightarrow\rightarrow X$ would be a surjective function and X would be finite. Now we can select an element of set $X - f(\{1, 2, ... n-1\})$ and accept f(n) to be this element. Therefore, utilising the principle of induction we have defined the function f for all $n \in Z_+$.

It can be noted that the function f is injective. Let $m < n$. Then f(m) belongs to the set f({l n — 1}), whereas f(n), by definition, does not. Therefore, $f(n) \neq f(m)$.

3.4 AXIOM OF CHOICE

Axiom of Choice is a very important and interesting axiom which is extensively utilised in mathematics. The Axiom can be stated in many ways. It is also true that many seemingly unrelated statements on closer analysis appear to be equivalent to it.

Definition. Let X be a non-empty collection of non-empty and disjoint sets, then there exists a set Y consisting of exactly one element from each element of X. In other words, a set Y such that Y is contained in the union of the elements of X, and for each $X \in X$, the set $X \cap Y$ has only one element.

Another way of stating this axiom is that for any family X of nonempty and disjoint sets, there exists a set that consists of exactly one element from each element of X.

Example 1

Let X be the set of countries in the world. Now each country can be considered as a set with the cities of that country as its elements. Then the union of X is the set of all cities in the world. A function which picks up the capital city and matches it with the country is a choice function. Similarly the function that matches the most densely populated city or the largest city of each country with its country is also a choice function.

Example 2

Let S be a set which has all pairs of shoes in the world as its elements. Then there exists a function that picks the left shoe out of each pair. This is a choice function for S.

It may be observed that in the examples quoted above and in any other example that can be stated, the choice function can be defined as a rule that identifies the element to be selected, i.e. select the capital city, select the largest city, select the left shoe of the pair, in the above example. The rule that is defined

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seems to be fairly simple and straight forward no matter how dense or large the set might be. In fact no axiom is required for such decisions, a simple rule is adequate. Therefore, what the Axiom of Choice does by a consequence is that it ensures existence of a rule or function that will meet the Choice.

3.4.1 Existence of a Choice Function

Lemma. Let X be a non-empty collection of non-empty sets, then there exists a function f such that it selects one member of each of the sets which are elements of X.

$$f : X \rightarrow\rightarrow U X \text{ for all } X \in X, \text{ such that } f(X) \text{ is an element of } X$$

The function f is called choice function for X.

[Note the difference between this lemma and axiom of choice: sets are not required to be disjoint]

Let X be an element of X. Then let $X^I = \{(X,x) : x \in X)$.

In other words we have constructed X^I as a collection of ordered pairs, where the first element is the set X, and the second element is an element of X. That is the set X^I is a subset of the Cartesian product $X^I \bullet U X$ for all $X \in X^I$.

Since X is a collection of non-empty sets, means X has at least one element x. Therefore, X^I contains at least one element given as (X, x) implying it is non-empty. It can now be seen that if X^I_1 and X^I_2 are two different sets belonging to X then there corresponding elements (X^I_1, x_1) and (X^I_2, x_2) are different sice there co-ordinates are different.

Now let us define another collection of sets Y such that

$$Y = \{ X^I : X \in X];$$

Please note that Y by the definition above is a collection of disjoint nonempty subsets of

$$X^I \bullet U X \text{ for all } X \in X^I$$

Now from the choice axiom we know that there exists a set Y consisting of exactly one element from each element of Y. Now we need to prove that Y is the rule for the desired choice function.

It can be seen that Y is a subset of $X^I \bullet U X$ for all $X \in X^I$.

Further, Y contains exactly one element from each set X^I, therefore, for each $X \in X$, the set Y contains only one ordered pair (X, x). Therefore, Y is the rule for a function from the collection X to the set U X such that $X \in X$.

Also, if $(X, x) \in Y$, then x is an element of X, thus Y(X) is an element of X.

Check Your Progress

3. Define axiom of choice.
4. What is meant by partial order relation?
5. What do you mean by total order relation?

3.5 WELL ORDERED SETS

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Let us recapitulate from Unit 1, that a binary relation R on a set X is a subset of the product X × X. Also recollect that a relation will have many properties. We had introduced few properties in unit 1. We will recapitulate them here and add a few more which are relevant to our discussion. We may recall that a relation R, for ease of denoting, is written as x R y for all x, y ∈ X.

3.5.1 Properties

(a) Reflexivity: x R x is true for all x ∈ X

(b) Symmetry: if x R y then y R x for all x ∈ X

(c) Antisymmetry: if x R y and y R x then x = y for all x, y ∈ X

(d) Asymmetry: if x R y then not y R x for all x, y ∈ X

(e) Transitivity: if x R y and y R z then x R z

(f) Connex or Comparability: either x R y or y R x.

Recall from unit 1 that any relation which is reflexive, symmetric and transitive is called an equivalence relation. Now we define some more types.

A relation which is reflexive, transitive and antisymmetric is called a **partial order**. A relation which is antisymmetric, transitive and comparable is called to be a **total order**. It may be argued that comparability implies reflexivity. Therefore, total order is a special type of partial order but partial order need not be total order.

A set X along with a binary relation R is called **totally ordered** if the relation R is a total order. Similarly a set X along with a binary relation R is called **partially ordered** if R is a partial order.

Now let us consider a set Y which is a subset of a partial order X and $y \in Y$ is such that, for every element $y_1 \in Y$, $y \leq y_1$, then y is called the least element of Y.

A set X which is totally ordered set is called well-ordered if its each non-empty subset has a minimum element. It may be observed that every finite set with a total order is well-ordered.

Example 3

Let us consider the Cartesian product of Z+ with the set {A, B}

Z+ X {a, b} = {a1, a2, a3, ...b1, b2, b3..........}

Alphabetical ordering has been applied on the product set above. In accordance with the alphabetical ordering it can be assumed that the element that comes first, or that is to the left is smaller than the one which comes later or the one that is on the right. It can be noticed that each non-empty subset of the above set will have a smallest element.

Example 4

The set of positive integers is well ordered. However, set of integers is not well ordered because the subset containing negative integers does not have a smallest element.

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Example 5

Consider a set $A = Z_+ \times Z_+$. As in example 1 above, the set A can be represented by dictionary order of an infinite sequence of infinite sequences.

Consider set B to be a subset of A, and C be the subset of Z_+ such that its elements are all first coordinates of elements of set B. Now let c_0 be the smallest element of C. Then we have a set Y as a non-empty subset of Z+ given as:

$\{y : c_0 \times y\}$ for all y belonging to B

Now if y_0 is the smallest element of Y then by dictionary order $c_0 \times y_0$ is the smallest element of B. Therefore A is well ordered.

From the discussion above following can be summarised:-

(a) Every subset of a well-ordered set A is well-ordered for the restriction of order on A.

(b) If set X and Y are well-ordered sets, then $X \times Y$ is well-ordered in the dictionary order.

Theorem 2: Every nonempty finite ordered set has the order type of a section of the set of positive integers Z_+, so it is well ordered.

Proof: In order to prove the theorem we first show that every finite ordered set X has a smallest element.

If X has only one element, then that element is the smallest. Now, let it be true for a subset having m - 1 elements, then let set X have m number of elements and assume element x_0 belongs to X. Then the set $X - \{x_0\}$ has the smallest element as x_1 then the smaller of $\{x_0, x_1\}$ is the smallest element of X.

Next we prove that there exists an order-preserving bijection of X with $\{1, 2, ...m\}$ for all m. Once again if X has one element, this proof is trivial. Now let us assume that it is true for subsets having m-1 elements. Now assume x_m is the largest element of X. By definition there exists an order preserving bijection such that $g : X - \{x\} \rightarrow\rightarrow \{1, 2, ... m-1\}$

Define an order preserving bijection $f: X \rightarrow\rightarrow \{1, 2, ... m\}$ by letting

$$f(x) = g(x) \text{ for } x \neq x_m.$$
$$F(x_m) = m.$$

Hence, a finite ordered set has only one order type.

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3.6 WELL-ORDERING THEOREM

For any set X there exists an order relation on X which is well-ordering. This theorem is also called Zermelo's theorem and was proved by Zermelo in 1904. It is equivalent to Axiom of Choice. In Georg Cantor's **opinion the well-ordering theorem is unobjectionable principle of thought**. There has been protracted debate as to the correctness of the proof given by Zermelo. The concept of well ordering any arbitrary uncountable set without any positive procedure has been contested by many mathematicians. On close study of the proof given by Zermelo it was found that the proof could be suspected only on one point, that involving the choice axiom. As a consequence many scholars rejected the choice axiom. However, over the years mathematicians have come to accept the proof and the theorem.

Corollary: There exist an uncountable number of well-ordered sets.

Lemma:There exists a well ordered set X having a largest element L such that the section S_L of X by L is uncountable but every other section of X is countable.

Proof: Let us assume that set A is an uncountable and well-ordered set.

Let B = {1, 2} x A in the alphabetical order; then some section of B. Let L be the smallest element of B for which the section of B by is uncountable. Then let X have this set and L as its elements. It needs to be born in mind that S_L is an uncountable well-ordered set all sections of which are countable. It is called a minimal uncountable well-ordered set. One of the very important properties of this set is discussed in the next theorem.

Theorem 3: If X is a countable subset of S_L, then X has an upper bound in S_L.

Proof: Assume that set X is a countable subset of S_L. For each $x \in X$, the Section S_x is countable. Therefore, the union $Y = U S_x$ for all $x \in X$, is also countable. Now as S_L is uncountable, the set Y does not include all elements of S_L. Let p be an element of S_L that is not in Y. Then the element p is an upper bound for X, since if $p < x$ for any $x \in X$, then p belongs to S_x and hence to Y in contradiction of our selection.

3.7 THE MAXIMUM PRINCIPLE

It may be recalled that the axiom of choice leads to the theorem that every set can be well-ordered. The axiom of choice has other equivalences that are very important and of lot of consequence for mathematicians. These are collectively called as the Maximum Principles. In the early 20th Century mathematicians like M Zorn, S Bochner and K. Kuratowslu, used the well-ordering theorem to prove these. It is only later that it was understood that they were equivalent to the well-ordering theorem.

Prior stating the Theorem and its proof, we will need to define a partial order.

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Definition: For any set X, a relation R is called a strict partial order on X if R is Non-reflexive and Transitive. In other words:-

(a) If x R x is not true for all $x \in X$, and

(b) If x R y and y R z are true then x R z is true for all x, y and $z \in X$.

It may be noted that if R is a strict partial order on elements of X then it is commonly denoted as $\prec$. The above two properties are the same as that of a simple order minus the property of comparability. Therefore, if $\prec$ is a strict partial order on set X then there can be a subset of X whose all elements are comparable under $\prec$. Then this subset of X is simply ordered by the relation $\prec$.

3.7.1 The Maximum Principle Theorem

Let X be a set and assume $\prec$ is a strict partial order on X, then there exists a maximal simply ordered subset Y of X.

In order to explain this important theorem, we can put it in simple words as that for every set X having a strict partial order $\prec$, there is a simply ordered set Y which is subset of X; and there is no other subset of X which is simply ordered and is a superset of Y.

Example 6

Consider a set of concentric rings. These rings can be considered as a combination of sets. The outer most ring contains all the rings inside it. This is a maximal simply ordered collection.

Example 7

Rectangular blocks of a planned city containing sub-blocks are a collection of maximal simply ordered collection.

Proof: A simple explanation of the theorem and a logical construction to prove it would be given in this section. Assume your house containing all house hold goods as a set H. Now pick up any house hold item and keep it in an empty room R of the house. Now pick up another item from the house and compare it with the first one with respect to some comparison rule. If it is comparable then keep it in the room R with the earlier item else discard it. Using the same rule of comparison check all items of the house and all items that are comparable should be kept in R, the others are to be discarded. In the end we will have a room R filled with items (elements) that are comparable to each other by a specific rule and no other item belonging to your house H is an element of this room R because it is not comparable. Therefore, the room R is a simply ordered set which is a subset of H and it is maximal since no other subset of H under the same rule can be larger.

The above proof seems logical and understandable for smaller sets. However, in case of infinite sets the same needs to be discussed further. Let us take the case of countably infinite set to begin with.

Let the above set H be a countably infinite set. Now let us do an indexing function on all items of H bijectively with Z_+ so that $H = \{h_1, h_2, h_3, ...\}$. Indexing helps in nomenclature of the elements of the set H.

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Now let f be a function such that f: $H \rightarrow\rightarrow \{0, 1\}$. In other words the function f assigns value 0 to h_i if we put the house hold item in room R and it assigns value 1 to h_i if we discard the item. That is f(i) = 0 for all h_i comparable to other elements. Implies:

$$\{h_j : j < i \text{ and } f(j) = 0\}$$

It can be seen from definition of recursive principle that the above equation is a unique function f: $H \rightarrow\rightarrow \{0, 1\}$ and that the set of those h_j for which f(j) is equal to zero is maximal.

Now in the case where set H is not countable then we would need to use the well-ordering theorem to prove. In this case let the items of set H be indexed with the elements of a well-ordered set X such that

$H = \{h_i : \in X\}$. In accordance with the well-ordering theorem, we know that there exists a bijection between H and some well-ordered set X. Now, the proof is simple and can be progressed as done in the case of countably infinite case.

3.7.2 Maximal Element

Consider a set X and let $\prec$ be a strict partial order on X. If Y is a subset of X, an upper bound on Y is an element x of X such that for every y in Y, either y = x or $y \prec c$. A maximal element of X is an element m such that for no element x of X, $m \prec x$ is true.

3.8 ZORN'S LEMMA

Consider a set X that is strictly partially ordered. If every simply ordered subset of X has an upper bound in X, then X has a maximal element.

Proof: Zorn's lemma can be easily understood in relation with the maximum principle; i.e for a set X maximum principle implies that X has a maximal simply ordered subset Y. The hypothesis of Zorn's lemma tells us that Y has an upper bound m in X. The element m is then automatically a maximal element of X. For if $m \prec n$ for some element n of X, then the set Y U {n}, which properly contains Y, is simply ordered because $y \prec n$ for every $b \in Y$. This fact contradicts maximality of Y.

Check Your Progress

6. State well-ordering theorem.
7. What is a partial order?
8. Define the maximum principle theorem.
9. What do you understand by Zorn's lemma?

3.9 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

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1. If a set has an infinite number of elements it is an infinite set.
2. A few examples of infinite sets are given below:

   (a) A set X comprising of all points in a plane is an infinite set.

   (b) Z = {……… -2, -1, 0, 1, 2, ……….} i.e. set of all integers is an infinite set.

   (c) Set of all positive integers which are multiple of 5 is an infinite set.
3. Axiom of choice states that for any family X of nonempty and disjoint sets, there exists a set that consists of exactly one element from each element of X.
4. A relation which is reflexive, transitive and antisymmetric is called a partial order.
5. A relation which is antisymmetric, transitive and comparable is called to be a total order.
6. Well-ordering theorem states that for any set X there exists an order relation on X which is well-ordering.
7. For any set X, a relation R is called a strict partial order on X if R is Non-reflexive and Transitive.
8. Let X be a set and assume $\prec$ is a strict partial order on X, then there exists a maximal simply ordered subset Y of X.
9. Consider a set X that is strictly partially ordered. If every simply ordered subset of X has an upper bound in X, then X has a maximal element.

3.10 SUMMARY

- Let X be a non-empty collection of non-empty sets, then there exists a function f such that it selects one member of each of the sets which are elements of X.
- A relation which is reflexive, transitive and antisymmetric is called a partial order.
- A relation which is antisymmetric, transitive and comparable is called to be a total order.
- Every subset of a well-ordered set A is well-ordered for the restriction of order on A.
- There exist an uncountable number of well-ordered sets.
- If X is a countable subset of S_L, then X has an upper bound in S_L
- If set X and Y are well-ordered sets, then X x Y is well-ordered in the dictionary order.
- Every nonempty finite ordered set has the order type of a section of the set of positive integers Z_+, so it is well ordered.

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- For any set X there exists an order relation on X which is well-ordering.
- The axiom of choice has other equivalences that are very important. These are collectively called as the Maximum Principles.
- For any set X, a relation R is called a strict partial order on X if R is Non-reflexive and Transitive.
- Zorn's Lemma states that if every simply ordered subset of a strictly partially ordered set X has an upper bound in X, then X has a maximal element.

3.11 KEY WORDS

- **Partial order:** For any set a relation is called a strict partial order on the set if the relation is non-reflexive and transitive.
- **Axiom of choice:** It states that for any family of no-empty and disjoint sets, there exists a set that consists of exactly one element from each element of that family.
- **Totally ordered set:** A set along with a binary relation is known as totally ordered if the relation is a total order.
- **Partially ordered set:** A set along with a binary relation is known as partially ordered if the relation is a partial order.
- **Well-ordering theorem:** It states that for any set there exists an order relation on the set which is well-ordering.
- **The maximum principle theorem:** Let X be a set and assume p is a strict partial order on X, then there exists a maximal simply ordered subset Y of X.
- **Zorn's lemma:** Consider a set X that is strictly partially ordered. If every simply ordered subset of X has an upper bound in X, then X has a maximal element.

3.12 SELF ASSESSMENT QUESTIONS AND EXERCISES

Short Answer Questions

1. Describe the infinite set theorem.
2. Explain well-ordering theorem.
3. Write a short note on the maximum principle.
4. Show that if X is an infinite set and x belongs to X, then X – {x} is also infinite.
5. If Y a subset of X is infinite, is X infinite?
6. Prove that if Y is an infinite set then Y x Y is also infinite?
7. Write a choice function for following sets:
   - Set X which is a collection of nonempty subsets of Z_+.

- Set Y which is a collection of nonempty subsets of Z.
- Set A which is a collection of nonempty subsets of Q where Q is a set of all rational numbers.
- Set B which is a collection of nonempty subsets of N where N is a set of all Natural numbers.

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8. Write a short note on Zorn's lemma.

Long Answer Questions

1. Explain axiom of choice giving suitable examples.
2. Describe well-ordered sets with suitable examples.
3. Discuss the maximum principle theorem with examples.
4. Show that if function f: $X \rightarrow\rightarrow Y$ is injective and X is not empty, then the function f has a left inverse.
5. Prove that if every subset of a countable set is countable.
6. Prove that if X_1, X_2, X_3 .... X_n are countable then $\{X_1 \text{ x } X_2 \text{ x } .... X_n\}$ is countable.
7. Prove that union of a finite number of sets that are countable is countable.
8. Show that the well-ordering theorem implies the choice axiom.
9. Prove that a well-ordered set satisfies least upper bound property.
10. Given two sets X and Y show using the well-ordering theorem that either X and Y have same cardinality, or one has cardinality lesser than the other.
11. Show that Zorn's Lemma implies Kuratowski lemma which states that if X is a collection of sets and if for every sub-collection Y of X that is simply ordered by proper inclusion, the union of the elements of Y is contained in X. Then X has an element that is properly contained in no other element of Y.

3.13 FURTHER READINGS

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NOTES

BLOCK - II

TOPOLOGICAL SPACES

UNIT 4 TOPOLOGICAL SPACES: AN INTRODUCTION

Structure

4.0 INTRODUCTION

A topological space may be defined as a set of points, along with a set of neighbourhoods for each point, satisfying a set of axioms relating points and neighbourhoods. Topological spaces are mathematical structures that authorize the formal definition of concepts such as continuity, connectedness, and convergence. The branch of mathematics that studies topological spaces is known as point-set topology or general topology. In this unit you will define a topological space in different equivalent ways and understand it with examples. You will understand countable complement topology, continuous functions topological constructions. You will analyse topological spaces with algebraic structure and order structure giving their classification. You will describe Euclidean spaces. Spectral, specialization preorder, Schwartz inequality and Minkowskis inequality are also discussed in this unit.

4.1 OBJECTIVES

After going through this unit, you will be able to:

- Define a topological space in various equivalent ways
- Understand a topological space with examples

- Comprehend continuous function, quotient space, homeomorphic space, and topological constructions
- Analyse topological spaces with algebraic structure and order structure giving their classification
- Define spectral order and canonical preorder
- Prove and understand Schwartz inequality and Minkowskis inequality to describe Euclidean spaces

4.2 TOPOLOGICAL SPACES

The word *topology* is used for a family of sets having definite properties that are used to define a topological space, a basic object of topology. *Topological spaces* are mathematical structures that authorize the formal definition of concepts such as convergence, connectedness and continuity. Hence, the branch of mathematics that studies topological spaces is called topology.

Definition

A topological space is a set X together with τ, a collection of subsets of X, satisfying the following axioms:

1. The empty set and X are in τ.
2. τ is closed under arbitrary union.
3. τ is closed under finite intersection.

The collection τ is called a **topology** on X. The elements of X are usually called *points*, though they can be any mathematical objects. A topological space in which the *points* are functions is called a function space. The sets in τ are called the open sets and their complements in X are called closed sets. A subset of X may be neither closed nor open, either closed or open, or both. A set that is both closed and open is called a clopen set. The following are examples of topological sets:

- $X = \{1, 2, 3, 4\}$ and collection $\tau = \{\{\ \}, \{1, 2, 3, 4\}\}$ of only the two subsets of X required by the axioms form a topology, the trivial topology (indiscrete topology).
- $X = \{1, 2, 3, 4\}$ and collection $\tau = \{\{\ \}, \{2\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}, \{1, 2, 3, 4\}\}$ of six subsets of X forms a topology.
- $X = \{1, 2, 3, 4\}$ and collection $\tau = P(X)$ (the power set of X) form a third topology, the discrete topology.
- $X = \boldsymbol{Z}$, the set of integers and collection τ equal to all finite subsets of the integers plus $\boldsymbol{Z}$ itself is not a topology, because the union of all finite sets not containing zero is infinite but is not all of $\boldsymbol{Z}$, and so is not in τ.

Equivalent Definitions

There are various additional equivalent ways to define a topological space. Consequently, each of the following defines a category equivalent to the category of topological spaces above. For example, using De Morgan's laws, the axioms defining open sets above become axioms defining closed sets:

1. The empty set and X are closed.
2. The intersection of any collection of closed sets is also closed.
3. The union of any pair of closed sets is also closed.

Using these axioms, another way to define a topological space is as a set X together with a collection τ of subsets of X satisfying the following axioms:

1. The empty set and X are in τ.
2. The intersection of any collection of sets in τ is also in τ.
3. The union of any pair of sets in τ is also in τ.

Under this definition, the sets in the topology τ are the closed sets and their complements in X are the open sets. Topological space can also be defined using the Kuratowski closure axioms, which define the closed sets as the fixed points of an operator on the power set of X. A neighbourhood of a point x is any set that has an open subset containing x. The *neighbourhood system* at x consists of all neighbourhoods of x. A topology can be determined by a set of axioms concerning all neighbourhood systems.

Various types of topologies can be placed on a set to form a topological space. When every set in a topology τ_1 is also in a topology τ_2, we say that τ_2 is better than τ_1 and τ_1 is coarser than τ_2. A proof that relies only on the existence of certain open sets will also hold for any better topology and similarly a proof that relies only on certain sets not being open applies to any coarser topology. The collection of all topologies on a given fixed set X forms a complete lattice: if $F = \{\tau_{á} : \alpha \text{ in } A\}$ is a collection of topologies on X, then the meet of F is the intersection of F and the join of F is the meet of the collection of all topologies on X that contain every member of F.

Continuous Functions

A function between topological spaces is called **continuous** if the inverse image of every open set is open. A homeomorphism is a bijection that is continuous and whose inverse is also continuous. Two spaces are called *homeomorphic* if there exists a homeomorphism between them. From the standpoint of topology, homeomorphic spaces are essentially identical.

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4.2.1 Examples of Topological Spaces

A given set may have many different topologies. If a set is given a different topology, it is viewed as a different topological space. Any set can be given the discrete topology in which every subset is open. The only convergent sequences in this topology are those that are eventually constant. Also, any set can be given the trivial topology also termed as the indiscrete topology in which only the empty set and the whole space are open. Every sequence in this topology converges to every point of the space. This example shows that in general topological spaces, limits of sequences need not be unique. However, often topological spaces must be Hausdorff spaces where limit points are unique.

There are many ways of defining a topology on **R**, the set of real numbers. The standard topology on **R** is generated by the open intervals. The set of all open intervals forms a base or basis for the topology, meaning that every open set is a union of some collection of sets from the base. In particular, this means that a set is open if there exists an open interval of non zero radius about every point in the set. More generally, the Euclidean spaces $\mathbf{R}^n$ can be given a topology. In the usual topology on $\mathbf{R}^n$ the basic open sets are the open balls. Similarly, **C** and $\mathbf{C}^n$ have a standard topology in which the basic open sets are open balls. Every metric space can be given a metric topology, in which the basic open sets are open balls defined by the metric. This is the standard topology on any normed vector space. On a finite dimensional vector space this topology is the same for all norms.

Many sets of linear operators in functional analysis are endowed with topologies that are defined by specifying when a particular sequence of functions converges to the zero function. Any local field has a topology native to it and this can be extended to vector spaces over that field. Every manifold has a natural topology since it is locally Euclidean. Similarly, every simplex and every simplicial complex inherits a natural topology from $\mathbf{R}^n$.

The Zariski topology is defined algebraically on the spectrum of a ring or an algebraic variety. On $\mathbf{R}^n$ or $\mathbf{C}^n$, the closed sets of the Zariski topology are the solution sets of systems of polynomial equations. A linear graph has a natural topology that generalizes many of the geometric aspects of graphs with vertices and edges. The Sierpiñski space is the simplest non-discrete topological space. It has important relations to the theory of computation and semantics.

There exist numerous topologies on any given finite set. Such spaces are termed as finite topological spaces. Any set can be given the cofinite topology in which the open sets are the empty set and the sets whose complement is finite. This is the smallest $\mathbf{T}_1$ topology on any infinite set.

Any set can be given the cocountable topology, in which a set is defined as open if it is either empty or its complement is countable. When the set is uncountable,

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this topology serves as a counterexample in many situations. The **cocountable topology** or **countable complement topology** on any set X consists of the empty set and all cocountable subsets of X, i.e., all sets whose complement in X is countable. It follows that the only closed subsets are X and the countable subsets of X. Every set X with the cocountable topology is Lindelöf, since every nonempty open set omits only countably many points of X. It is also $\mathbf{T}_1$, as all singletons are closed. The only compact subsets of X are the finite subsets, so X has the property that all compact subsets are closed, even though it is not Hausdorff if uncountable.

The real line can also be given the lower limit topology. Here, the basic open sets are the half open intervals $[a, b)$. This topology on $\mathbf{R}$ is strictly finer than the Euclidean topology defined above; a sequence converges to a point in this topology if and only if it converges from above in the Euclidean topology. This example shows that a set may have many distinct topologies defined on it.

If Γ is an ordinal number, then the set $\Gamma = [0, \Gamma)$ may be endowed with the order topology generated by the intervals (a, b), $[0, b)$ and (a, Γ) where a and b are elements of Γ.

Topological Constructions

Every subset of a topological space can be given the subspace topology in which the open sets are the intersections of the open sets of the larger space with the subset. For any indexed family of topological spaces, the product can be given the product topology, which is generated by the inverse images of open sets of the factors under the projection mappings. For example, in finite products, a basis for the product topology consists of all products of open sets. For infinite products, there is the additional requirement that in a basic open set, all but finitely many of its projections are the entire space.

A quotient space is defined as follows:

If X is a topological space and Y is a set, and if $f : X \to Y$ is a surjective function, then the quotient topology on Y is the collection of subsets of Y that have open inverse images under f. In other words, the quotient topology is the finest topology on Y for which f is continuous. A common example of a quotient topology is when an equivalence relation is defined on the topological space X. The map f is then the natural projection onto the set of equivalence classes.

The Vietoris topology on the set of all non-empty subsets of a topological space X, named for Leopold Vietoris, is generated by the following basis:

For every n-tuple $U_1, ..., U_n$ of open sets in X, we construct a basis set consisting of all subsets of the union of the U_i that have non-empty intersections with each U_i.

4.2.2 Classification of Topological Spaces

Topological spaces can be generally classified up to homeomorphism by their topological properties. A topological property is a property of spaces that is invariant under homeomorphisms. To prove that two spaces are not homeomorphic find a topological property not shared by them. Examples of such properties include connectedness, compactness and various separation axioms.

Topological Spaces with Algebraic Structure

For any algebraic objects we can introduce the discrete topology, under which the algebraic operations are continuous functions. For any such structure that is not finite, we often have a natural topology compatible with the algebraic operations in the sense that the algebraic operations are still continuous. This leads to concepts such as topological groups, topological vector spaces, topological rings and local fields.

Topological Spaces with Order Structure

- **Spectral:** A space is spectral if and only if it is the prime spectrum of a ring (Hochster theorem).
- **Specialization preorder:** In a space the **specialization** (or **canonical**) **preorder** is defined by $x \le y$ if and only if $\mathrm{cl}\{x\} \subseteq \mathrm{cl}\{y\}$. Here cl denotes canonical preorder.

Euclidean Spaces

For each $n \in N$, let R denote the set of all ordered n-tuples $x = (x_1, x_2,..., x_n)$, where, $x_1,..., x_n$ are real numbers, called the coordinates of x. The elements of $\mathbf{R}^n$ will be called points or vectors and will always be denoted by letters, a, b, c, x, y, z, etc. Let $y = (y_1, y_2,..., y_n)$ and let a be a real number. We define the addition of vectors and multiplication of a vector by a real number (called scalar) as follows:

$$x + y = (x_1 + y_1,..., x_n + y_n);\ \alpha x = (\alpha\, x_1,...,\alpha\, x_n).$$

These definitions show that $x + y \in \mathbf{R}^n$ and $\alpha x \in \mathbf{R}^n$. It is easy to see that for these operations the commutative, associative and distributive laws hold and thus $\mathbf{R}^n$ is a vector space over the real field $\mathbf{R}$. The zero element of $\mathbf{R}^n$ (usually called the origin or the null vector) is the point 0, all of whose coordinates are zero.

The scalar product or inner product of two vectors x and y is defined by,

$$x, y = \sum_{i=1}^{n} x_i y_i,$$

Also the norm of x is defined by $|x| = (x, x)^{1/2} = \left(\sum_{i=1}^{n} x_i^{\,2}\right)$

The vector space $\mathbf{R}^n$ with the preceding inner product and norm is called Euclidean n-space.

In the sequel, we shall need the following inequalities.

(i) If $z_1, z_2, ..., z_n$ are complex numbers (or in particular real numbers), then

$$|z_1 + z_2 + \ldots + z_n| \leq |z_1| + |z_2| + \ldots + |z_n|.$$

(ii) If z_i, ω_i $(i = 1,...,n)$ are complex numbers, then

$$\left|\sum_{i=1}^{n} z_i \overline{\omega_i}\right| \leq \left(\sum_{i=1}^{n} |z_i|^2\right)^{1/2} \left(\sum_{i=1}^{n} |\omega_i|^2\right)^{1/2} \qquad \textbf{(Schwarz Inequality)}$$

If a_i, $b_i (i = 1,...,n)$ are real numbers, then the above inequality

reduces to $\left|\sum_{i=1}^{n} a_i b_i\right| \leq \left(\sum_{i=1}^{n} a_i^{\,2}\right)^{1/2} \left(\sum_{i=1}^{n} b_i^2\right)^{1/2}$

(iii) If z_i, $\omega_i (i = 1,...,n)$ are complex numbers, then

$$\left[\sum_{i=1}^{n} |z_i + \omega_i|^2\right]^{1/2} \left[\sum_{i=1}^{n} |z_i|^2\right]^{1/2} + \left[\sum_{i=1}^{n} |\omega_i|^2\right]^{1/2} \qquad \textbf{(Minkowskis Inequality)}$$

Theorem 1: Let $x, y, z \in \mathbf{R}^n$ and let α be real. Then

(i) $|x| \geq 0$, (ii) $|x| = 0$ if $x = 0$,

(iii) $|\alpha x| = |\alpha|\ |x|$, (iv) $|x, y| \leq |x| |y|$,

(v) $|x + y| \leq |x| + |y|$, (vi) $|x - y| \leq |x - y| + |y - z|$.

Proof: The proofs of cases (i), (ii) and (iii) are trivial but obvious. To prove (iv), by Schwarz inequality, we have

$$\left|\sum_{i=1}^{n} x_i y_i\right| \leq \left[\sum_{i=1}^{n} x_i^{\,2}\right]^{1/2} \left[\sum_{i=1}^{n} y_i^{\,2}\right]^{1/2}$$

Hence, by the definition of scalar product and norm, we at once obtain $|x.y| \leq |x|\ |y|$.

Again using case (iv), we have

$|x + y|^2 = (x + y) \,.\, (x + y) = x.x + 2x.y + y.y$

$\leq |x|^2 + 2|x|\,|y| + |y|^2 = (|x| + |y|)^2$

so that $|x + y| \leq |x| + |y|$. Thus case (v) is proved.

Finally, case (vi) follows from case (v) by replacing x by $x - y$ and y by $y - z$.

4.3 PROBLEMS

NOTES

Following examples will help you understand the concept of topological space. Consider that if (X, τ) is a topological space then a set $A \subseteq X$ is said to be open if $A \in \tau$ and A is said to be closed if $A^c \in \tau$. Furthermore, if A is both open and closed, then we say that A is clopen. The following examples will help you to identify the open, closed and clopen sets of a topological space (X, τ).

Example 1. Let $X = \{a, b, c, d\}$ and consider that the topology,

$\tau = \{\varnothing, \{c\}, \{a, b\}, \{c, d\}, \{a, b, c\}, X\}$

What are the open, closed, and clopen sets of X with respect to this topology space?

Solution: We find the solution as follows.

The open sets of X are those sets which form τ:

Open sets of $X = \{\varnothing, \{c\}, \{a, b\}, \{c, d\}, \{a, b, c\}, X\}$

The closed sets of X are the complements of all of the open sets:

Closed sets of $X = \{\varnothing, \{a, b, d\}, \{c, d\}, \{a, b\}, \{d\}, X\}$

The clopen sets of X are the sets that are both open and closed:

Clopen sets of $X = \{\varnothing, \{a, b\}, \{c, d\}, X\}$

Example 2. Prove that if X is a set and every $A \subseteq X$ is clopen with respect to the topology τ then τ is the discrete topology on X.

Solution. Let X be a set and let every $A \subseteq X$ be clopen. Then every $A \subseteq X$ is open, i.e., every subset of X is open.

Therefore, $\tau = P(X)$.

Hence, τ is the discrete topology on X.

Example 3. Consider the topological space $(\mathbb{Z}, \tau)$ where τ is the co-finite topology.

(1) Determine whether the set of even integers is open, closed, and/or clopen.

(2) Determine whether the set $\mathbb{Z} \setminus \{1, 2, 3\}$ is open, closed, and/or clopen.

(3) Determine whether the set $\{-1, 0, 1\}$ is open, closed, and/or clopen.

Show that any nontrivial subset of $\mathbb{Z}$ is never clopen.

Solution: The co-finite topology τ is described by:

$$\tau = \{U \subseteq X : U = \varnothing \text{ or } U^c \text{ is finite}\}$$

We first consider the set of even integers which we denote by,

$E = \{..., -2, 0, 2, ...\}$

Then the E^c is considered as the set of odd integers, i.e.,

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$E^c = \{..., -3, -1, 1, 3, ...\}$ which is an infinite set. Therefore $E \notin \tau$ so E is not open.

Furthermore, we have that $(E^c)^c = E$ is an infinite set and $E^c \notin \tau$ so E is not closed either.

Now consider the set $\mathbb{Z}\backslash\{1, 2, 3\}$. We have that $(\mathbb{Z}\backslash\{1, 2, 3\})^c = \{1, 2, 3\}$ which is a finite set. Therefore $\mathbb{Z}\backslash\{1, 2, 3\} \in \tau$, so $\mathbb{Z}\backslash\{1, 2, 3\}$ is open.

Now consider the complement $(\mathbb{Z}\backslash\{1, 2, 3\})^c = \{1, 2, 3\}$. The complement of this set is $(\{1, 2, 3\})^c = \mathbb{Z}\backslash\{1, 2, 3\}$ which is an infinite set.

Therefore, $(\{1, 2, 3\})^c \notin \tau$.

Hence, $\mathbb{Z}\backslash\{1, 2, 3\}$ is not closed.

Now we consider the set $\{-1, 0, 1\}$.

We have $(\{-1, 0, 1\})^c = \mathbb{Z}\backslash\{-1, 0, 1\}$ which is an infinite set.

Therefore, $\{-1, 0, 1\} \notin \tau$ so $\{-1, 0, 1\}$ is not open.

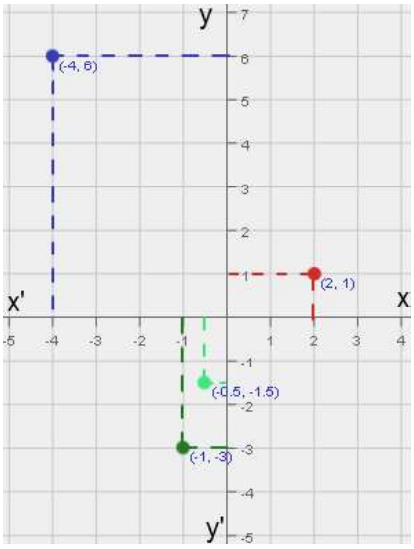
Now consider the complement $(\{-1, 0, 1\})^c = \mathbb{Z}\backslash\{-1, 0, 1\}$. The complement of this set is $\{-1, 0, 1\}$ is finite, thus $(\{-1, 0, 1\})^c \in \tau$. Hence, $\{-1, 0, 1\}$ is closed.

Finally, let $A \subseteq \mathbb{Z}$ be a nontrivial subset of $\mathbb{Z}$, i.e., $A \neq \varnothing$ and $A \neq \mathbb{Z}$. Suppose that A is clopen. Then A is both open and closed. Hence by definition, A and A^c are both open. Hence A^c and A are both finite. But $\mathbb{Z} = A \cap A^c$ which implies that $\mathbb{Z}$ is a finite set - which is absurd since the set of integers is an infinite set. Hence A cannot be clopen.

Example 4. Draw the following points in Euclidean two-dimensional space:

(2, 1), (–1, –3), (–0.5, –1.5) and (–4, 6)

Solution: The above mentioned points can be drawn in Euclidean two-dimensional space as shown in the figure below of Euclidean Graph



NOTES

Check Your Progress

1. What do you understand by a topological space?
2. Define a function space.
3. What is a continuous space?
4. What is meant by homeomorphic space?
5. What is a spectral space?
6. Give definition for a specialization preorder.

4.4 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1. A topological space is a set X together with t, a collection of subsets of X, satisfying the following axioms:
   - The empty set and X are in τ.
   - τ is closed under arbitrary union.
   - τ is closed under finite intersection.

   The collection τ is called a **topology** on X.
2. A topological space in which the *points* are functions is called a function space.
3. A function between topological spaces is called **continuous** if the inverse image of every open set is open.
4. A homeomorphism is a bijection that is continuous and whose inverse is also continuous. Two spaces are called *homeomorphic* if there exists a homeomorphism between them.
5. A space is spectral if and only if it is the prime spectrum of a ring (Hochster theorem).
6. In a space the **specialization** (or **canonical**) **preorder** is defined by $x \leq y$ if and only if $\text{cl}\{x\} \subseteq \text{cl}\{y\}$. Here cl denotes canonical preorder.

4.5 SUMMARY

- The word topology is used for a family of sets having definite properties that are used to define a topological space, a basic object of topology.
- Topological spaces are mathematical structures that authorize the formal definition of concepts such as continuity, connectedness, and convergence.

- A topological space in which the points are functions is called a function space.
- A function between topological spaces is called continuous if the inverse image of every open set is open.
- The countable topology or countable complement topology on any set X consists of the empty set and all countable subsets of X, i.e., all sets whose complement in X is countable.
- The quotient topology is the finest topology on Y for which f is continuous.
- A space is said to be spectral if and only if it is the prime spectrum of a ring.
- The vector space with the preceding inner product and norm is known as Euclidean n-space.

4.6 KEY WORDS

- **Function space:** A topological space in which the points are functions is called a function space.
- **Continuous function:** A function between topological spaces is called continuous if the inverse image of every open set is open.
- **The countable topology:** The countable topology or countable complement topology on any set X consists of the empty set and all countable subsets of X, i.e., all sets whose complement in X is countable.
- **The Sierpinski space:** It is the simplest non-discrete topological space.
- **Spectral space:** A space is said to be spectral if and only if it is the prime spectrum of a ring.

4.7 SELF ASSESSMENT QUESTIONS AND EXERCISES

Short Answer Questions

1. What is a topological space? Give various equivalent ways to define a topological space.
2. Give brief account of topological constructions.
3. Analyse topological spaces giving their classification.

Long Answer Questions

1. Give a detailed account of topological spaces giving suitable examples.
2. Explain the concept of topological constructions with the help of examples.
3. Explain Euclidean spaces using Schwartz inequality and Minkowskis inequality.

4.8 FURTHER READINGS

NOTES

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UNIT 5 ORDER TOPOLOGY

NOTES

Structure

5.0 INTRODUCTION

In this unit, you will learn about product topological spaces, nets and filters. Product topology is the topology on the Cartesian product $X \times Y$ of two topological spaces whose open sets are the unions of subsets $A \times B$, where A and B are open subsets of X and Y, respectively. This definition extends in a natural way to the Cartesian product of any finite number n of topological spaces. The product topology is also called Tychonoff topology. An order topology is a certain topology that can be explained on any totally ordered set. If X is a totally ordered set, the order topology on X is generated by the subbase of "open rays". A topological space X is known as orderable if there exists a total order on its elements such that the order topology induced by that order and the given topology on X coincide. Nets are used in general topology and analysis to instill non-metrizable topological spaces with convergence properties. Filters are also used to define convergence and are a generalization of nets.

5.1 OBJECTIVES

After going through this unit, you will be able to:

- Define product topological spaces and projection mappings
- Describe Tychonoff product topology in terms of standard subbase and its characterization
- Explain separation axioms and product spaces
- Explicate local connectedness and compactness of product spaces
- State product space as first axiom space

- Define nets and filters
- Describe convergence of nets and filters

5.2 THE ORDER TOPOLOGY

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In mathematics, an **order topology** is a certain topology that can be defined on any totally ordered set. It is a natural generalization of the topology of the real numbers to arbitrary totally ordered sets.

Definition. If X is a totally ordered set, then the order topology on X is generated by the following given sub-base:

$$(a, \infty) = \{x \mid a < x\}$$
$$(-\infty, b) = \{x \mid x < b\}$$

for all a, b in X. This is equivalent to the open intervals as,

$$(a, b) = \{x \mid a < x < b\}$$

This forms a base for the order topology.

Additionally, a topological space X is called orderable if there exists a total order on its elements such that the order topology induced by that order and the given topology on X coincide. The order topology makes X into a completely normal Hausdorff space. The standard topologies on R, Q, Z and N are the order topologies.

Definition. A topology defined on a totally ordered set X whose open sets are all the finite intersections of subsets of the form,

$$\{x \in X \mid x > a\}$$

Or, $$\{x \in X \mid x < a\}$$

Where $a \in X$.

The order topology of the real line is the Euclidean topology. The order topology of $\mathbb{N}$ is the discrete topology, since for all $n \in \mathbb{N}$,

$\{n\} = \{x \in \mathbb{N} \mid x > n-1\} \bigcap \{x \in \mathbb{N} \mid x < n+1\}$ is an open set.

Definition. If X is a simply ordered set, then there is a standard topology for X, defined using the order relation termed as 'Order Topology'.

Assume that X is a set having order relation <. Given $a < b \in X$, then there are four subsets of X termed as intervals determined by a and b as follows:

$$(a, b) = \{x \mid a < x < b\}$$
$$(a, b] = \{x \mid a < x \leq b\}$$
$$[a, b) = \{x \mid a \leq x < b\}$$
$$[a, b] = \{x \mid a \leq x \leq b\}$$

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5.3 THE PRODUCT TOPOLOGY ON X × Y

In topology and related areas of mathematics, a product space is defined to be the Cartesian product of a family of topological spaces equipped with a natural topology called the product topology. The box topology agrees with the product topology when the product is over only finitely many spaces. Wherein the product topology makes the product space a categorical product of its factors, the box topology is too fine. Hence the product topology is considered natural.

Let X be the (possibly infinite) Cartesian product of the topological spaces X_i, indexed by $i \in \mathbf{I}$ such that,

$$X := \prod_{i \in \mathbf{I}} X_i$$

and the *canonical projections* $p_i : X \to X_i$, then the *product topology* on X is defined to be the coarsest topology, which means the topology with the fewest open sets, for which all the projections p_i are continuous. The product topology is sometimes termed as the *Tychonoff topology*. The open sets in the product topology are finite or infinite unions of sets of the form $\prod U_i$, where each U_i is open in X_i and $U_i \neq X_i$ only finitely many times.

The product topology on X is the topology generated by sets of the form $p_i^{-1}(U)$, where i is in $\mathbf{I}$ and U is an open subset of X_i. We can also say that, the sets $\{p_i^{-1}(U)\}$ form a subbase for the topology on X. A subset of X is open iff it is a possibly infinite union of intersections of finitely many sets of the form $p_i^{-1}(U)$. The $p_i^{-1}(U)$ are sometimes called open cylinders and their intersections are cylinder sets. We can describe a basis for the product topology using bases of the constituting spaces X_i. A basis consists of sets ΠU_i, where for all but finitely many i, $U_i = X_i$, and otherwise it is a basic open set of X_i. Particularly, for a finite product for the product of two topological spaces, the products of base elements of the X_i give a basis for the product ΠX_i.

As a general rule, the product of the topologies of each X_i forms a basis for the box topology on X. Although in general, the box topology is finer than the product topology, but for finite products they coincide. If we start with the standard topology on the real line $\mathbf{R}$ and define a topology on the product of n copies of $\mathbf{R}$ in this manner, then we obtain the ordinary Euclidean topology on $\mathbf{R}^n$. The Cantor set is homeomorphic to the product of countably many copies of the discrete space $\{0,1\}$ and the space of irrational numbers is homeomorphic to the product of countably many copies of the natural numbers, where again each copy carries the discrete topology.

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Let (X_1, T_1) and (X_2, T_2) be topological spaces and let Y be their Cartesian product, $Y = X_1 \times X_2 = \{(x_1, x_2): x_1 \in X_1, x_2 \in X_2\}$. The sets of the form $U \times V$, with $U \in T_1$ and $V \in T_2$, form a base and, since $U \times V = U \times X_2 \cap X_1 \times V$, the sets $\{U \times X_2, X_1 \times V : U \in T_1, V \in T_2\}$ constitute a subbase for the product topology.

The projection maps, p_1 and p_2, on the Cartesian product $X_1 \times X_2$, are defined by

$p_1 : X_1 \times X_2 \to X_1, (x_1, x_2) \to x_1$

$p_2 : X_1 \times X_2 \to X_1, (x_1, x_2) \to x_2.$

Then the product topology is the weakest topology on the Cartesian product $X_1 \times X_2$ such that both p_1 and p_2 are continuous – the $\sigma(X_1, X_2, \{p_1, p_2\})$-topology.

We would like to generalize this to an arbitrary Cartesian product of topological spaces. Let $\{(X_\alpha, T_\alpha): \alpha \in \mathbf{I}\}$ be a collection of topological spaces indexed by the set $\mathbf{I}$. We recall that

$X = \Pi_\alpha X_\alpha$, the Cartesian product of the X_α's, is defined to be the collection of maps γ from $\mathbf{I}$ into the union $\bigcup_\alpha X_\alpha$ satisfying $\gamma(\alpha) \in X_\alpha$ for each $\alpha \in \mathbf{I}$. We can think of the value $\gamma(\alpha)$ as the α-coordinate of the point γ in X. The idea is to construct a topology on $X = \Pi_\alpha X_\alpha$ built from the individual topologies T_α. There are two possibilities. The first is the weakest topology on X with respect to which all the projection maps $p_\alpha \to X_\alpha$ are continuous. The second is to construct the topology on X whose open sets are unions of 'super rectangles', that is, sets of the form $\Pi_\alpha U_\alpha$, where $U_\alpha \in T_\alpha$ for every $\alpha \in \mathbf{I}$. In general, these two topologies are not the same.

Definition: The product topology, denoted T_{prod} on the Cartesian product of the topological spaces $\{(X_\alpha, T_\alpha): \alpha \in \mathbf{I}\}$ is the $\sigma(\Pi_\alpha X_\alpha, \mathcal{F})$-topology, where $\mathcal{F}$ is the family of projection maps $\{p_\alpha : \alpha \in \mathbf{I}\}$.

Note: Let G be a non empty open set in X, equipped with the product topology and let $\gamma \in G$. Then, by definition of the topology, there exist $\alpha_1, \ldots., \alpha_n \in \mathbf{I}$ and open sets $U_{\alpha i}$ in $X_{\alpha i}$, $1 \le i \le n$, such that

$$\gamma \in p_{\alpha 1}^{-1}(U_{\alpha 1}) \cap \ldots.. \cap p_{\alpha 1}^{-1}(U_{\alpha 1}) \subseteq G .$$

Hence there are open sets S_α, $\alpha \in \mathbf{I}$, such that $\gamma \in \Pi_\alpha S_\alpha \subseteq G$ and where $S_\alpha = X_\alpha$ except possibly for at most a finite number of values of α in $\mathbf{I}$. This means that G can differ from X in at most a finite number of components.

Now let us consider the second candidate for a topology on X. Let $\mathcal{S}$ be the topology on X with base given by the sets of the form $\Pi_\alpha V_\alpha$, where $V_\alpha \in T_\alpha$ for $\alpha \in \mathbf{I}$. Thus, a non empty set G in X belongs to $\mathcal{S}$ if and only if for any point x in G there exist $V_\alpha \in T_\alpha$ such that

$X \in \Pi_\alpha V_\alpha \subseteq G$.

Here there is no requirement that all but a finite number of the V_α are equal to the whole space X_α.

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Definition: The topology on the Cartesian product $\Pi_\alpha X_\alpha$ constructed in this way is called the box-topology on X and is denoted T_{box}.

In general, $\mathcal{S}$ is strictly finer than the product topology, T_{prod}.

Theorem 1: A net (x_λ) converges in $(\Pi_\alpha X_\alpha, T_{\text{prod}})$ if and only if $(p_\alpha(x_\lambda))$ converges in (X_α, T_α) for each $\alpha \in \mathbf{I}$.

Theorem 2: Suppose that $\mathcal{F}$ is any collection of subsets of a given set X satisfying the finite intersection property. Then there is a maximal collection $\mathcal{D}$ containing $\mathcal{F}$ and satisfying the finite intersection property, i.e., if $\mathcal{F} \subseteq \mathcal{F}$ and if $\mathcal{F}$ satisfies the finite intersection property, then $\mathcal{F} \subseteq \mathcal{D}$. Furthermore,

i. If $A_1, \ldots, A_n \in \mathcal{D}$, then $A_1 \cap \ldots \cap A_n \in \mathcal{D}$, and

ii. If A is any subset of X such that $A \cap D \neq \phi$ for all $D \in \mathcal{D}$, then $A \in \mathcal{D}$.

Proof: As might be expected, we shall use Zorn's lemma. Let $\mathcal{C}$ denote the collection of those families of subsets of X which contain $\mathcal{F}$ and satisfy the finite intersection property. Then $\mathcal{F} \in \mathcal{C}$, so $\mathcal{C}$ is not empty. Evidently, $\mathcal{C}$ is ordered by set-theoretic inclusion. Suppose that ϕ is a totally ordered set of families in $\mathcal{C}$. Let $A = \bigcup_{\mathcal{S} \in \varphi} \mathcal{S}$. Then $\mathcal{F} \subseteq A$, since $\mathcal{F} \subseteq \mathcal{S}$, for all $\mathcal{S} \in \phi$. We shall show that A satisfies the finite intersection property .To see this, let $S_1, \ldots, S_n \in A$. Then each S_i is an element of some family $\mathcal{S}_i$ that belongs to ϕ. But ϕ is totally ordered and so there is i_0 such that $\mathcal{S}_i \subseteq \mathcal{S}_{i_0}$ for all $1 \leq i \leq$ n. Hence $S_1 \cap \ldots \cap S_n \neq \phi$ since $\mathcal{S}_{i_0}$ satisfies the finite intersection property. It follows that A is an upper bound for ϕ in $\mathcal{C}$. Hence, by Zorn's lemma, $\mathcal{C}$ contains a maximal element, $\mathcal{D}$, say.

i. Now suppose that $A_1, \ldots, A_n \in \mathcal{D}$ and let $B = A_1 \cap \ldots \cap A_n$. Let $\mathcal{D}' = \mathcal{D} \cup \{B\}$. Then any finite intersection of members of $\mathcal{D}'$ is equal to a finite intersection of members of $\mathcal{D}$. Thus $\mathcal{D}'$ satisfies the finite intersection property. Clearly, $\mathcal{F} \subseteq \mathcal{D}'$, and so, by maximality, we deduce that $\mathcal{D}' = \mathcal{D}$. Thus $B \in \mathcal{D}$.

ii. Suppose that $A \subseteq X$ and that $A \cap \mathcal{D} \neq \phi$ for every $D \in \mathcal{D}$. Let $\mathcal{D}' = \mathcal{D} \cup \{A\}$, and let $D_1, \ldots, D_m \in \mathcal{D}'$. If $D_i \in \mathcal{D}$, for all $1 \leq i \leq m$, then $D_1 \cap \ldots \cap D_m \neq \phi$ since $\mathcal{D}$ satisfies the finite intersection property. If $D_i \in A$ and some $D_j \neq A$, then $D_1 \cap \ldots \cap D_m$ has the form $D_1 \cap \ldots \cap D_k \cap A$ with $D_1 \cap \ldots \cap D_k \in \mathcal{D}$. By (i), $D_1 \cap \ldots \cap D_k \in \mathcal{D}$ and so, by hypothesis, $A \cap (D_1 \ldots \cap D_k) \neq \phi$. Hence $\mathcal{D}'$ satisfies the finite intersection property and, again by maximality, we have $\mathcal{D}' = \mathcal{D}$ and thus $A \in \mathcal{D}$.

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We are now ready to prove Tychonov's theorem which states that the product of compact topological spaces is compact with respect to the product topology.

Theorem 3 (Tychonov's theorem): Let $\{(X_\alpha, T_\alpha): \alpha \in \mathbf{I}\}$ be any given collection of compact topological spaces. Then the Cartesian product $(\Pi_\alpha X_\alpha, T_{\text{prod}})$, equipped with the product topology is compact.

Proof: Let $\mathcal{F}$ be any family of closed subsets of $\Pi_\alpha X_\alpha$ satisfying the finite intersection property. We must show that $\bigcap_{F\in\mathcal{F}} F \neq \phi$. By the previous Theorem 4.2, there is a maximal family $\mathcal{D}$ of subsets of $\Pi_\alpha X_\alpha$ satisfying the finite intersection property and with $\mathcal{F} \subseteq \mathcal{D}$ (Note that the members of $\mathcal{D}$ need not all be closed sets).

For each $\alpha \in \mathbf{I}$, consider the family $\{p_\alpha(D): D \in \mathcal{D}\}$. Then this family satisfies the finite intersection property because $\mathcal{D}$ does. Hence $\{\overline{p_\alpha(D)} : D \in \mathcal{D}\}$ satisfies the finite intersection property. But this is a collection of closed sets in the compact space (X_α, T_α), and so $\bigcap_{D\in\mathcal{D}} \overline{p_\alpha(D)} \neq \phi$.

That is, there is some $x_\alpha \in X_\alpha$ such that $x_\alpha \in \overline{p_\alpha(D)}$ for every $D \in \mathcal{D}$. Let $x \in \Pi_\alpha X_\alpha$ be given by $p_\alpha(x) = x_\alpha$, i.e., the αth coordinate of x is x_α. Now, for any $\alpha \in \mathbf{I}$ and for any $D \in \mathcal{D}$, $x_\alpha \in \overline{p_\alpha(D)}$ implies that for any neighbourhood U_α of x_α we have $U_\alpha \cap p_\alpha(D) \neq \phi$. Hence $p^{-1}_\alpha(U_\alpha) \cap D \neq \phi$ for every $D \in \mathcal{D}$. By the previous Theorem 4.2, it follows that $p_\alpha^{-1}(U_\alpha) \in \mathcal{D}$.

Hence, again by the previous Theorem 4.2, for any $\alpha_1,\ldots, \alpha_n \in \mathbf{I}$ and neighbourhoods $U_{\alpha_1},\ldots,U_{\alpha_n}$ of $x_{\alpha_1},\ldots, x_{\alpha_n}$, respectively, $p_{\alpha_1}^{-1}(U_{\alpha_1}) \cap,\ldots, p_{\alpha_n}^{-1}(U_{\alpha_n}) \in \mathcal{D}$.

Furthermore, since $\mathcal{D}$ has the finite intersection property, we have that $p_{\alpha_1}^{-1}(U_{\alpha_1}) \cap,\ldots, p_{\alpha_n}^{-1}(U_{\alpha_n}) \cap \mathrm{D} \neq \phi$.

For every finite family $\alpha_1,\ldots, \alpha_n \in \mathbf{I}$ neighbourhoods $U_{\alpha_1},\ldots,U_{\alpha_n}$ of $x_{\alpha_1},\ldots, x_{\alpha_n}$, respectively, and every $D \in \mathcal{D}$.

We shall show that $x \in \overline{D}$ for every $D \in \mathcal{D}$. To see this, let G be any neighbourhood of x. Then, by definition of the product topology, there is a finite family $\alpha_1,\ldots, \alpha_m \in \mathbf{I}$ and open sets $U_{\alpha_1},\ldots,U_{\alpha_m}$ such that

$$x \in p_{\alpha_1}^{-1}(U_{\alpha_1}) \cap,\ldots, p_{\alpha_m}^{-1}(U_{\alpha_m}) \subseteq G.$$

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But we have shown that for any $D \in \mathcal{D}$,

$D \cap p_{\alpha_1}^{-1}(U_{\alpha_1}) \cap, \ldots, p_{\alpha_m}^{-1}(U_{\alpha_m}) \neq \phi$ and therefore $D \cap G \neq \phi$. We deduce that $x \in \overline{D}$, the closure of D, for any $D \in \mathcal{D}$. In particular, $x \in \overline{F} = F$ for every $F \in \mathcal{F}$. Thus, $\bigcap_{F \in \mathcal{F}} F \neq \phi$ since it contains x. The result follows.

5.3.1 Projection Mappings

In mathematics, in general, a *projection* is a mapping of a set or of a mathematical structure which is idempotent, i.e., a projection is equal to its composition with itself. A *projection* may also refer to a mapping which has a left inverse. Both these ideas are related. Let p be an idempotent map from a set E into itself (thus $p^\circ p = \mathrm{Id}_E$) and $F = p(E)$ be the image of p. If we denote the map p viewed as a map from E onto F by π and the injection of F into E by i, then we get $i^\circ\pi = \mathrm{Id}_F$. On the other hand, $i^\circ\pi = \mathrm{Id}_F$ implies that $\pi^\circ i$ is idempotent.

Initially, the notion of projection was introduced in Euclidean geometry to denote the projection of the three-dimensional Euclidean space onto a plane. The two main projections of this kind are

- **Central projection:** It is the projection from a point onto a plane. If C is the center of projection, then the projection of a point P distinct from C is the intersection with the plane of the line CP. The point C and the points P such that the line CP is parallel to the plane do not have any image by the projection.
- The **projection onto a plane parallel to a direction** D: The image of a point P is the intersection with the plane of the line parallel to D passing through P.

Definition: For each $\beta \in \Lambda$, the mapping $\pi_\beta : \underset{\lambda}{\pi} X_\lambda \rightarrow X_\beta$ assigning to each element $<x_\lambda>$ of $\underset{\lambda}{\pi} X_\lambda$ its βth coordinate, $\underset{\beta}{\pi}(< x_\lambda >) = x_\beta$ is known as the projection mapping associated with the index β.

Consider the set $\underset{\beta}{\pi}^{-1}(G_\beta)$ where G_β is an open subset of X_β. It consists of all points $p = \{a_\lambda ; \lambda \in \Lambda\}$ in $\underset{\lambda}{\pi} X_\lambda$ such that $\underset{\beta}{\pi}(p) \in G_\beta$. In other words, $\underset{\beta}{\pi}^{-1}(G_\beta) = \underset{\lambda}{\pi} Y_\lambda$ where $Y_\beta = G_\beta$ and $Y_\lambda = X_\lambda$ whenever $\lambda \neq \beta$ that is $\pi_\beta^{-1}(G_\beta) = X_1 \times X_2 \times \ldots \times X_{\beta-1} \times G_\beta \times X_{\beta+1} \ldots . \times \ldots$

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Definition: For each λ in an arbitrary index set Λ, let (X_λ, T_λ) be a topological space and let $X = \underset{\lambda}{\pi} X_\lambda$. Then the topology T for X, which has a subbase the collection $B_* = \{\pi_\beta^{-1}(G_\lambda); \lambda \in \wedge \text{ and } G_\lambda \in T_\lambda\}$ is called the product topology or Tichonov topology for X and (X, T) is called the product space of the given spaces.

The collection B_* is called the defining subbase for T. The collection B of all finite intersection of elements of B_* would then form the base for T.

Note: The projection mappings are continuous for G_β in T_β–open in $X_\beta \Rightarrow \underset{\beta}{\pi}^{-1}(G_\beta)$ $\in B_*$ which is a subbase for T and therefore $\underset{\beta}{\pi}^{-1}(G_\beta)$ is T-open in $X = \underset{\lambda}{\pi} X_\lambda$.

Theorem 4: Let A be a member of the defining base for a product space $X = \underset{\lambda}{\pi} X_\lambda$. Then the projection of A into any coordinate space is open.

Proof: Since A belongs to the defining base for X.

$A = \underset{\lambda}{\pi}\{X_\lambda; \lambda \neq \alpha_1, \alpha_{2,\ldots\ldots} \alpha_m\} \times G_{\alpha_1} \times \ldots \times G_{\alpha_m}$ where G_{α_i} is an open subset of X_{α_i} . So for any projection $\pi_\beta : X \to X_\beta$,

$$\pi_\beta(A) = \begin{cases} X_\beta & \text{if } \beta \neq \{\alpha_1, \alpha_2, \ldots, \alpha_m\} \\ G_\beta & if \ \beta \in \{\alpha_1, \alpha_2, \ldots, \alpha_m\} \end{cases}$$

In either case $\pi_\beta(A)$ is an open set.

Theorem 5: Every projection $\pi_\beta : X \to X_\beta$ on a product space $X = \underset{\lambda}{\pi} X_\lambda$ is open.

Let G be an open subset of X. For every point $p \in G$, there is a member A of the defining base of the product topology such that $p \in A \subset G$. Thus for any projection $\pi_\beta : X \to X_\beta, p \in G \Rightarrow \pi_\beta(P) \in \pi_\beta(A) \subset \pi_\beta(G)$.

But $\pi_\beta(A)$ is an open set. Therefore, every point $\pi_\beta(P)$ in $\pi_\beta(G)$ belongs to an open set $\pi_\beta(A)$ which is contained in $\pi_\beta(G)$ is an open set.

Note: As each projection is continuous and open, but projections are not closed maps, e.g., consider the space $\mathbf{R} \times \mathbf{R}$ with product topology.

Let $H = \{(x, y); x, y \in \mathbf{R} \text{ and } xy = 2\}$

Here H is closed in $\mathbf{R} \times \mathbf{R}$ but $\pi_1(H) = \mathbf{R} \sim (0)$

is not closed with respect to the usual topology for $\mathbf{R}$ where π_1 is the projection in the first coordinate space $\mathbf{R}$.

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5.3.2 Tychonoff Product Topology in Terms of Standard Subbase and its Characterization

In topology, a *subbase* for a topological space X with topology $\mathbf{T}$ is a subcollection B of $\mathbf{T}$ which generates $\mathbf{T}$, such that $\mathbf{T}$ is the smallest topology containing B. Following are some useful equivalent formulations of the definition:

Let X be a topological space with topology $\mathbf{T}$. A subbase of this topology $\mathbf{T}$ is defined as a subcollection B of $\mathbf{T}$ satisfying one of the two following equivalent conditions:

1. The subcollection B generates the topology $\mathbf{T}$. This implies that $\mathbf{T}$ is the smallest topology containing B and so any topology U on X containing B must also contain $\mathbf{T}$.
2. The collection of open sets consisting of all finite intersections of elements of B, together with the set X and the empty set, forms a basis for $\mathbf{T}$. This means that every non-empty proper open set in $\mathbf{T}$ can be written as a union of finite intersections of elements of B. Clearly, given a point x in a proper open set U, there are finitely many sets $S_1, \ldots, S_n$ of B, such that the intersection of these sets contains x and is contained in U.

For any subcollection S of the power set $P(X)$, there is a unique topology having S as a subbase. Particularly, the intersection of all topologies on X containing S satisfies this condition. In general, however, there is no unique subbasis for a given topology. Thus, we can start with a fixed topology and find subbases for that topology. We can also start with an arbitrary subcollection of the power set $P(X)$ and form the topology generated by that subcollection. Although, we can freely use either equivalent definition above but in many cases, one of the two conditions is more useful than the other.

Occasionally, a somewhat different definition of subbase is given which requires that the subbase B cover X. In this case, X is an open set in the topology generated, because it is the union of all the $\{B_i\}$ as B_i ranges over B. This means that there can be no confusion regarding the use of nullary intersections in the definition. However, with this definition, the two definitions above do not remain equivalent. In other words, there exist spaces X with topology $\mathbf{T}$, such that there exists a subcollection B of $\mathbf{T}$ such that $\mathbf{T}$ is the smallest topology containing B, yet B does not cover X. In practice, this is a rare occurrence. For example, a subbase of a space satisfying the $\mathbf{T}_1$ separation axiom must be a cover of that space.

Let us consider an example first.

Example 1: Consider the topology $T = \{\phi, X, (a),(b, c)\}$ on $X = (a, b, c)$ and the topology $T^* = \{\phi, Y, (u)\}$ on $Y = (u, v)$. Determine the defining subbase B_* of the product topology on $X \times Y$.

Solution:

$X \times Y = \{(a, u), (a, v), (b, u), (b, v), (c, u), (c, v)\}$ is the product set on which the product topology is defined. The defining subbase B_* is the class of inverse sets $\pi_x^{-1}(G)$ and $\pi_y^{-1}(H)$ where G is an open subset of X and H is an open subset of Y. Computing, we have

$$\pi_x^{-1}(X)\,\pi_y^{-1}(Y) = X \times Y,$$

$$\pi_x^{-1}(\varphi) = \pi_x^{-1}(\varphi) = \phi$$

$$\pi_x^{-1}(a) = \{(a, u), (a, v)\}$$

$$\pi_x^{-1}(b,c) = \{(b, u), (b, v), (c, u), (c, v)\}$$

$$\pi_y^{-1}(u) = \{(a, u), (b, u), (c, u)\}$$

Hence, the defining subbase B_* consists of the subsets of $X \times Y$ above. The defining base B consists of finite intersections of members of the defining subbase, that is,

$$B = \{\phi, X \times Y, (a, u), (b, u), (c, u)\}$$
$$\{(a, u), (a, v)\}, \{(b, u), (b, v), (c, u), (c, v)\}$$
$$\{(a, u), (b, u), (c, u)\}$$

Theorem 6: Let (X_λ, T_λ) be an arbitrary collection of topological spaces and let $X = \underset{\lambda}{\pi} X_\lambda$. Let T be a topology for X. If T is the product topology for X, then T is the smallest topology for X for which projections are continuous and conversely also.

Proof: Let π_λ be the λ-th projection map and let G_i be any T_λ-open subset of X_λ. Then since T is the product topology for X, $\pi_\lambda^{-1}(G_\lambda)$ is a member of the subbase for T and hence $\pi_\lambda^{-1}(G_\lambda)$ must be T-open. It follows that π_λ is T- T_λ continuous. Now let V be any topology on X such that π_λ is V- T_λ continuous for each $\lambda \in \Lambda$.

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Then $\pi_\lambda^{-1}(G_\lambda)$ is V-open for every $G_\lambda \in T_\lambda$. Since V is a topology for X, V contains all the unions of finite intersections of members of the collection $\{\pi_\lambda^{-1}(G_\lambda)$; $\lambda \in \Lambda$ and $G_\lambda \in T_\lambda\}$.

$\Rightarrow$ V contains T that is T is coarser than V. Thus T is the smallest topology for X such that π_λ is T– T_λ continuous for each $\lambda \in \Lambda$.

Conversely, let B_* be the collection of all sets of the form $\pi_\lambda^{-1}(G_\lambda)$ where G_λ is an open subset of X_λ for $\lambda \in \Lambda$. Then by definition, a topology V for X will make all the projections π_λ continuous if and only if $B_* \subset V$. Thus the smallest topology for X which makes all the projections continuous, is the topology determined by B_* as a subbase.

Theorem 7: A function $f: Y \to X$ from a topological space Y into a product space $X = \underset{\lambda}{\pi} X_\lambda$ is continuous if and only if for every projection $\pi_\beta : X \to X_\beta$, the composition mapping $\pi_\beta o f: Y \to X_\beta$ is continuous.

Proof: By the definition of product space, all projections are continuous. So if f is continuous, then π_β of being the composition of two continuous functions, is also continuous.

On the other hand, suppose every composition function $\pi_\beta o f: Y \to X_\beta$ is continuous. Let G be an open subset of X_β. Then by the continuity of $\pi_\beta o f$, $(\pi_\beta o f)^{-1}(G) = f^{-1}[\pi_\beta^{-1}(G)]$ is an open set in Y. But the class of sets of the form $\pi_\beta^{-1}(G)$ where G is an open subset of X_β is the defining subbase for the product topology on X. Since their inverse under f are open subsets of Y, f is a continuous function.

Note: The projection π_x and π_y of the product of two sets X and Y are the mappings of $X \times Y$ onto X and Y respectively defined by setting $\pi_x (<x, y>) = x$ and $\pi_y (<x, y>) = y$.

Theorem 8: If X and Y are topological spaces, the family of all sets of the form $V \times W$ with V open in X and W open in Y is a base for a topology for $X \times Y$.

Proof: Since the set $X \times Y$ is itself of the required form, $X \times Y$ is the union of all the members of the family. Now let $<x, y> \in (V_1 \times W_1) \cap (V_2 \times W_2)$ with V_1 and V_2 open in X and W_1 and W_2 open in Y. Then $<x, y> \in (V_1 \cap V_2) \times (W_1 \times W_2) = (V_1 \times W_1) \cap (V_2 \times W_2)$ with $V_1 \cap V_2$ open in X and $W_1 \cap W_2$ open in Y. Then the family is a base for topology for $X \times Y$.

Theorem 9: Let C_i be a closed subset of a space X_i for $i \in \mathbf{I}$. Then $\underset{i \in I}{\pi} C_i$ is a closed subset of $\underset{i \in I}{\pi} X_i$ with respect to the product topology.

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Proof: Let $X = \underset{i \in I}{\pi} X_i$ and $C = \underset{i \in I}{\pi} C_i$. We claim $X - C$ is an open set in the product topology on X. Let $x \in X - C$. Then $C = \bigcap_{i \in I} \pi_i^{-1}(C_i)$ and so $x \notin C$ implies that there exists $j \in \mathbf{I}$ such that $\pi_j(x) \notin C_j$. Let $V_j = X_j - C_j$ and let $V = \pi_j^{-1}(V_j)$. Then V_j is an open subset of X_j and so V is an open subset (in fact a member of the standard subbase) in the product topology on X. Evidently $\pi_j(x) \in V_j$ and so $x \in V$. Moreover, $C \cap V = \phi$ since $\pi_j(C) \cap \pi_j(V) = \phi$. So $V \subset X - C$. Thus, $X - C$ is a neighbourhood of each of its point. Therefore, $X - C$ is open and C is closed in X.

Check Your Progress

1. What do you mean by a product space?
2. Give definition for the product topology.
3. What does the Tychonov's theorem state?
4. What do you understand by a projection?
5. What is a subbase in topology?

5.4 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1. A product space is defined to be the Cartesian product of a family of topological spaces equipped with a natural topology called the product topology.
2. The product topology, denoted T_{prod} on the Cartesian product of the topological spaces $\{(X_\alpha, T_\alpha): \alpha \in \mathbf{I}\}$ is the $\sigma(\Pi_\alpha X_\alpha, \mathcal{F})$-topology, where $\mathcal{F}$ is the family of projection maps $\{p_\alpha : \alpha \in \mathbf{I}\}$.
3. Let $\{(X_\alpha, T_\alpha): \alpha \in \mathbf{I}\}$ be any given collection of compact topological spaces. Then the Cartesian product $(\Pi_\alpha X_\alpha, T_{\text{prod}})$, equipped with the product topology is compact.
4. A projection is a mapping of a set or of a mathematical structure which is idempotent, i.e., a projection is equal to its composition with itself. A projection may also refer to a mapping which has a left inverse. Both these ideas are related.
5. In topology, a subbase for a topological space X with topology **T** is a subcollection B of **T** which generates **T**, such that **T** is the smallest topology containing B.

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5.5 SUMMARY

- Product topology is the topology on the Cartesian product X×Y of two topological spaces whose open sets are the unions of subsets A×B, where A and B are open subsets of X and Y, respectively.
- The product topology is known as Tychonoff topology also.
- An order topology is a certain topology that can be explained on any totally ordered set.
- A topological space X is known as orderable if there exists a total order on its elements such that the order topology induced by that order and the given topology on X coincide.
- A product space is defined as the Cartesian product of a family of topological spaces equipped with a natural topology called the product topology.
- The Cantor set is homeomorphic to the product of countably many copies of the discrete space {0, 1}.
- A net (x_λ) converges in $(\Pi_\alpha X_\alpha, T_{prod})$ if and only if $(p_\alpha(x_\lambda))$ converges in (X_α, T_α) for each $\alpha \in \mathbf{I}$.
- If $\mathcal{F} \subseteq \mathcal{F}$ and if $\mathcal{F}$ satisfies the finite intersection property, then $\mathcal{F} \subseteq \mathcal{D}$.
- Every projection $\pi_\beta : X \rightarrow X_\beta$ on a product space $X = \underset{\lambda}{\pi} X_\lambda$ is open.
- For any subcollection S of the power set $P(X)$, there is a unique topology having S as a subbase.
- A function $f : Y \rightarrow X$ from a topological space Y into a product space $X = \underset{\lambda}{\pi} X_\lambda$ is continuous if and only if for every projection $\pi_\beta : X \rightarrow X_\beta$, the composition mapping $\pi_\beta o f : Y \rightarrow X_\beta$ is continuous.
- If X and Y are topological spaces, the family of all sets of the form $V \times W$ with V open in X and W open in Y is a base for a topology for $X \times Y$.

5.6 KEY WORDS

- **Product topology:** It is the topology on the Cartesian product X×Y of two topological spaces whose open sets are the unions of subsets A×B, where A and B are open subsets of X and Y, respectively.
- **Product space:** It is defined to be the Cartesian product of a family of topological spaces equipped with a natural topology called the product topology.

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- **Tychonov's theorem:** It states that the product of compact topological spaces is compact with respect to the product topology.
- **Projection:** it is a mapping of a set or of a mathematical structure which is idempotent, i.e., a projection is equal to its composition with itself.
- **Central projection:** Central projection is the projection from a point onto a plane.
- **Subbase:** In topology, a subbase for a topological space X with topology T is a sub collection B of T which generates T, such that T is the smallest topology containing B.

5.7 SELF ASSESSMENT QUESTIONS AND EXERCISES

Short Answer Questions

1. Write a short note on order topology.
2. Give a brief account of product topology.
3. Describe the Tychonov's theorem briefly.
4. Discuss projection mappings using suitable theorems.

Long Answer Questions

1. Explain the product topology on $X \times Y$.
2. Discuss Tychonov's product topology in terms of standard subbase and its characterization with the help of examples.

5.8 FURTHER READINGS

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NOTES

UNIT 6 TOPOLOGY: SUBSPACE, CLOSED SETS AND HAUSDORFF SPACES

Structure

6.0 INTRODUCTION

A subspace of a topological space X is a subset S of X which is equipped with a topology induced from that of X called the subspace topology. It is known as the relative topology, or the induced topology, or the trace topology also. The Kuratowski closure axioms are referred as a set of axioms that are used to define a topology on a set. In this unit, you will learn about the subspace topology, relative topology and topological groups. You will analyse topology in terms of Kuratowski closure operator and neighbourhood systems interpreting Kuratowski closure axioms. You will describe dense subsets, open sets, closed sets, limit point and Neighbourhoods. You will deduce Bolzano-Weierstrass theorem. You will understand Hausdorff spaces.

6.1 OBJECTIVES

After going through this unit, you will be able to:

- Understand subspace and relative topology, topological spaces and topological groups

- Analyse topology in terms of Kuratowski closure operator and neighbourhood systems interpreting Kuratowski closure axioms
- Comprehend dense subsets, open sets, closed sets, limit point and Neighbourhoods
- Prove and understand Bolzano-Weierstrass theorem
- Describe Hausdorff spaces

6.2 THE SUBSPACE TOPOLOGY

Consider a set X and a proper subset, $\phi \neq A \subset X$. Then $\{\phi, A, X\}$ is a topology containing A. Actually, it is the smallest topology. Further, suppose $\phi \neq B \neq A \subset X$. The smallest possible topology that contains both A, B is clearly $\{\phi, A, B, A \cap B, A \cup B, X\}$. Similarly, if there are several different sets $A_1, \ldots, A_n$ in the topology, many other sets must be also there. This is the idea behind generating a topology.

Definition: Let $(X, \mathbf{T})$ be a topological space. A set $\mathcal{B} \subset \mathbf{T}$ is called a base for $\mathbf{T}$ if,

$$T = \{\cup A : A \subset B\}$$

Any $G \in \mathcal{B}$ is called a basic open set. We will obtain the topology T by taking all arbitrary unions of sets in $\mathcal{B}$.

Let us consider the following collection of subsets in $\mathbf{R}$,

$$S = \{(a, \infty) : a \in \mathbf{R}\} \cup \{(-\infty, b) : b \in \mathbf{R}\}.$$

It is not a base for the standard topology of $\mathbf{R}$, because the intersection of two such subsets may not be in itself. However, these finite intersections are all the open intervals and hence they form a base. This becomes the typical example for a subbase.

Definition : Let (X, T) be a topological space. A set $S \subset \mathbf{T}$ is called a subbase if $\{S_1 \cap \ldots\ldots \cap S_n : S_j \in S, n \in \mathbf{Z}\}$ is a base for $\mathbf{T}$. Equivalently, $\mathbf{T} = \bigcup \{B \subset X : \mathcal{B} \bigcap \mathcal{A}$ for some finite subset $\mathcal{A} \subset S\}$.

Here, we can also say that $\mathbf{T}$ is generated by S. However, the same $\mathbf{T}$ may be generated by different subbases. A topological space $(X, \mathbf{T})$ may have different bases or subbases.

Theorem 1: Any nonempty collection $\mathbf{C} \subset \mathcal{P}(X)$ defines a topology $\mathbf{T}$ such that $\mathbf{C}$ is a subbase for $\mathbf{T}$.

Proof: Let $\mathcal{B} = \{\bigcap \mathcal{F} : \mathcal{F} \subset \mathbf{C}, \#(\mathcal{F}) < \infty\}$. Then, the topology is generated by $\mathbf{C}$, $\mathbf{T} = \{\bigcup \mathcal{A} : \mathcal{A} \subset \mathcal{B}\}$. In order to verify that $\mathbf{T}$ is a topology, use the De Morgan's law.

However, not every C can be a base; additional conditions are needed. The rest of the proof is left as an exercise.

6.2.1 Subspace and Relative Topology

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Topological Spaces: If X is a set, a family $\mathcal{U}$ of subsets of X defines a topology on X if,

(i) $\phi \in \mathcal{U}$, $X \in \mathcal{U}$.

(ii) The union of any family of sets in $\mathcal{U}$ belongs to $\mathcal{U}$.

(iii) The intersection of a finite number of sets in $\mathcal{U}$ belongs to $\mathcal{U}$.

If $\mathcal{U}$ defines a topology on X, then we say that X is a topological space. The sets in $\mathcal{U}$ are called open sets. The sets of the form $X\backslash U$, $U \in \mathcal{U}$, are called closed sets. If Y is a subset of X, then the closure of Y is the smallest closet set in X that contains Y.

Let Y be a subset of a topological space X. Then we may define a topology $\mathcal{U}_Y$ on Y, called the subspace or relative topology or the topology on Y induced by the topology on X, by taking

$$\mathcal{U}_Y = \{Y \cap U \mid U \in \mathcal{U}\}.$$

A system $\mathcal{B}$ of subsets of X is called a basis (or base) for the topology $\mathcal{U}$ if every open set is the union of certain sets in $\mathcal{B}$. Equivalently, for each open set U, given any point $x \in U$, there exists $B \in \mathcal{U}$ such that $x \in B \subset U$.

For example, the set of all bounded open intervals in the real line **R** forms a basis for the usual topology on **R**.

Let $x \in X$. A neighbourhood of x is an open set containing x. Let $\mathcal{U}_x$ be the set of all neighborhoods of x. A subfamily $\mathcal{B}_x$ of $\mathcal{U}_x$ is a basis or base at x, a neighborhood basis at x, or a fundamental system of neighbourhoods of x, if for each $U \in \mathcal{U}_x$, there exists $B \in \mathcal{B}_x$ such that $B \subset U$. A topology on X may be specified by giving a neighbourhood basis at every $x \in X$.

If X and Y are topological spaces, then there is a natural topology on the Cartesian product $X \times Y$ that is defined in terms of the topologies on X and Y, called the product topology. Let $x \in X$ and $y \in Y$. The sets $U_x \times V_y$ as U_x ranges over all neighbourhoods of x and V_y ranges over all neighbourhoods of y, forms a neighbourhood basis at the point $(x, y) \in X \times Y$, for the product topology.

If X and Y are topological spaces, a function $f: X \to Y$ is continuous if whenever U is an open set in Y, the set $f^{-1}(U) = \{x \in X | f(x) \in U\}$ is an open set in X. A function $f: X \to Y$ is a homeomorphism of X onto Y if f is bijective and both f and f^{-1} are continuous functions.

An **open covering** of a topological space X is a family of open sets having the property that every $x \in X$ is contained in at least one set in the family. A **subcover** of an open covering is an open covering of X which consists of sets belonging to the open covering. A topological space X is compact if every open covering of X contains a finite subcover.

A subset Y of a topological space X is compact if Y is compact in the subspace topology. A topological space X is locally compact if for each $x \in X$ there exists a neighbourhood of x whose closure is compact.

A topological space X is Hausdorff (or $\mathbf{T}_2$) if given distinct points x and $y \in X$, there exist neighbourhoods U of x and V of y such that $U \cap V = \phi$. A closed subset of a locally compact Hausdorff space is locally compact.

Topological Groups

A topological group G is a group that is also a topological space, having the property the maps $(g_1, g_2) \to g_1 g_2$ from $G \times G \to G$ and $g \to g^{-1}$ from G to G are continuous maps. In this definition, $G \times G$ has the product topology.

Lemma: Let G be a topological group. Then

(i) The map $g \to g^{-1}$ is a homeomorphism of G onto itself.

(ii) Fix $g_0 \in G$. The maps $g \to g_0 g$, $g \to g g_0$ and $g \to g_0 g g\ g_0^{-1}$ are homeomorphisms of G onto itself.

A subgroup H of a topological group G is a topological group in the subspace topology. Let H be a subgroup of a topological group G and let p: $G \to G/H$ be the canonical mapping of G onto G/H. We define a topology $U_{G/H}$ on G/H, called the quotient topology, by $\mathcal{U}_{G/H} = \{p(U) \mid U \in \mathcal{U}_G\}$. Here, $\mathcal{U}_G$ is the topology on G. By definition the canonical map p is open and continuous. If H is a closed subgroup of G, then the topological space G/H is Hausdorff. If H is a closed subgroup of G, then G/H is a topological group.

If G and G' are topological groups, a map f: $G \to G'$ is a continuous homomorphism of G into G' if f is a homomorphism of groups and f is a continuous function. If H is a closed normal subgroup of a topological group G, then the canonical mapping of G onto G/H is an open continuous homomorphism of G onto G/H.

A topological group G is a locally compact group if G is locally compact as a topological space.

Theorem 2: Let G be a locally compact group and let H be a closed subgroup of G. Then,

(i) H is a locally compact group in the subspace topology.

(ii) If H is normal in G, then G/H is a locally compact group.

(iii) If G' is a locally compact group, then $G \times G$ is a locally compact group in the product topology.

6.2.2 Topology in Terms of Kuratowski Closure Operator and Neighbourhood Systems

The Polish mathematician Kazimierz (Casimir) Kuratowski developed a radically different approach to specifying a topology for a set. Kuratowski considered particular functions from the set of subsets of **K** to the set of subsets of **K** as explained below.

A topology for a set **K** is a collection of subsets of **K** such that,

- The union of any arbitrary subcollection is also a member of the collection.
- The intersection of finite numbers of members of the collection is also a member of the collection.
- The null set belongs to the collection.
- The whole set **K** belongs to the collection.

The elements of the collection are called the open sets of the topology. The openness of a set is not a property of the set itself but it refers only to the membership of the set in the collection of subsets which is called the topology.

A set is defined as being closed with respect to a topology if its complement is open with respect to the topology; i.e., if its complement belongs to the topology. At least two sets, the null set and the whole set **K** are both open and closed in any topology of **K**.

If the closed sets of a topology are given the open sets can easily be constructed since they are simply the complements of the closed sets.

Kuratowski Closure Axioms

Every topological space consists of the following:

- A set of points.
- A class of subsets defined axiomatically as open sets.
- The set operations of union and intersection.

The class of open sets must be defined in such a way that the intersection of any finite number of open sets is itself open and the union of any infinite collection of open sets is likewise open. A point p is called a limit point of the set S if every open set containing p also contains some point (s) of S (points other than p, should p happen to lie in S). The concept of limit point is fundamental to topology and it can be used axiomatically to define a topological space by specifying limit points for each set according to rules known as the Kuratowski closure axioms. Any set of objects can be made into a topological space in various ways, but the usefulness of the concept depends on the manner in which the limit points are separated from each other.

The **Kuratowski closure axioms** are referred as a set of axioms that are used to define a topology on a set. They were first introduced by Kuratowski, in a slightly different form that applied only to Hausdorff spaces. In general topology, if X is a topological space and A is a subset of X, then the *closure* of A in X is defined to be the smallest closed set containing A or equivalently the intersection of all closed sets containing A. The *closure operator* C that assigns to each subset of A its closure $C(A)$ is thus a function from the power set of X to itself. The closure operator satisfies the following axioms:

1. **Isotonicity:** Every set is contained in its closure.
2. **Idempotence:** The closure of the closure of a set is equal to the closure of that set.
3. **Preservation of Binary Unions:** The closure of the union of two sets is the union of their closures.
4. **Preservation of Nullary Unions:** The closure of the empty set is empty.

Definition: An operator C of $\rho(X)$ into itself which satisfies the following four properties mentioned in theorem 4 is called a closure operator on the set X.

Theorem 3: In the topological space $(X, \mathbf{T})$, the closure operator has the following properties.

($\mathbf{K}_1$) $C(\phi) = \phi$

($\mathbf{K}_2$) $E \subseteq C(E)$

($\mathbf{K}_3$) $C\,(C(E)) = C(E)$

($\mathbf{K}_4$) $C\,(A \cup B) = C(A) \cup C(B)$

Proof: ($\mathbf{K}_1$): Because the void set is closed and also we know that a set A is closed if and only if,

$A = C(A)$, therefore it follows that $C(\phi) = \phi$.

($\mathbf{K}_2$): It follows from the definition as $C(E)$ is the smallest closed set containing E.

($\mathbf{K}_3$): Since $C(E)$ is the smallest closed set containing E, we have $C\,(C(E)) = C(E)$ by the result that a set is closed if and only if it is equal to its closure.

($\mathbf{K}_4$): Since $A \subset A \cup B$ and $B \subset A \cup B$, therefore

$C(A) \subset C\,(A \cup B)$ and $C(B) \subset C\,(A \cup B)$ and so

$$C(A) \cup C(B) \subset C\,(A \cup B) \qquad ...(1)$$

By ($\mathbf{K}_2$), we have

$A \subset C(A)$ and $B \subset C(B)$

Therefore, $A \cup B \subset C(A) \cup C(B)$

Since $C(A)$ and $C(B)$ are closed sets and so $C(A)\cup C(B)$ is closed. By the definition of closure, we have

$$C(A\cup B) \subset C[C(A)\cup C(B)] \qquad ...(2)$$

From Equations (1) and (2), we have

$$C(A\cup B) = C(A)\cup C(B).$$

Note: $C(A\cap B)$ may not be equal to $C(A)\cap C(B)$. For example, if $A = (0, 1)$, $B = (1, 2)$, then $C(A) = [0, 1]$, $C(B) = [1, 2]$.

Therefore,

$C(A)\cap C(B) = \{1\}$ where $A\cap B = \phi$

But $C(\phi) = \phi$. Therefore $C(A\cap B) = \phi$ and thus $C(A\cap B) \neq C(A)\cap C(B)$.

Note: The closed sets are simply the sets which are fixed under the closure operator.

Theorem 4: Let C^* be a closure operator defined on a set X. Let F be the family of all subsets F of X for which $C^*(F) = F$ and let **T** be a family of all complements of members of F. Then **T** is a topology for X and if C is the closure operator defined by the topology **T**. Then $C^*(E) = C(E)$ for all subsets $E \subseteq X$.

Proof: Suppose $G_\lambda \in \mathbf{T}$ for all λ. We must show that $\bigcup_\lambda G_\lambda \in \mathbf{T}$,i.e., $\left(\bigcup_\lambda G_\lambda\right)^C \in F$.

Thus, we must show that,

$$C^*\left[\left(\bigcup_\lambda G_\lambda\right)^C\right] = \left(\bigcup_\lambda G_\lambda\right)^C$$

By (K_2) $\left(\bigcup_\lambda G_\lambda\right)^C \subseteq C^*\left[\left(\bigcup_\lambda G_\lambda\right)^C\right]$

So we need only to prove that,

$$C^*\left[\left(\bigcup_\lambda G_\lambda\right)^C\right] \subseteq \left(\bigcup_\lambda G_\lambda\right)^C$$

By De Morgan's Law, this reduces to the form,

$$C^*\bigcap_\lambda\left[(G_\lambda)^C\right] \subseteq \bigcap_\lambda (G_\lambda)^C$$

Since $\left(\bigcap_\lambda (G_\lambda)^C\right) \subseteq \left((G_\lambda)^C\right)$ for each particular λ.

$C^*\left[\bigcap_\lambda (G_\lambda)^C\right] \subseteq C^*\left[(G_\lambda)^C\right]$ for each λ.

So, $C^*\left[\bigcap_\lambda (G_\lambda)^C\right] \subseteq \bigcap_\lambda C^*\left[(G_\lambda)^C\right]$

But, $G_\lambda \in \mathbf{T} \Rightarrow (G_\lambda)^C \in F$.

Hence, $C^*[(G_\lambda)^C] = (G_\lambda)^C$

Thus we have $C^*[\underset{\lambda}{\cap}(G_\lambda)^C] \subseteq \underset{\lambda}{\cap}(G_\lambda)^C$

Consequently, if $G_\lambda \in \mathbf{T}$, then $\underset{\lambda}{\cup} G_\lambda \in \mathbf{T}$.

To check that ϕ, $X \in \mathbf{T}$, we observe that by Kuratowski closure axiom ($\mathbf{K}_2$),

$X \subseteq C^*(X) \subseteq X \Rightarrow C^*(X) = X \Rightarrow X \in F$

Hence $X^C = \phi \in \mathbf{T}$.

Also by (Kurtowaski closure axiom $\mathbf{K}_1$) we have $C^*(\phi) = \phi \Rightarrow \phi \in F$

$\Rightarrow \phi^C = X \in \mathbf{T}$.

Finally consider that $G_1 G_2 \in \mathbf{T}$. Then by hypothesis,

$C^*(G_1)^C = G_1{}^C$ and $C^*(G_2)^C = G_2{}^C$

We may now calculate,

$$\begin{aligned} C^*[(G_1 \cap G_2)^C] &= C^*[\, G_1{}^C \cup G_2{}^C \,] \\ &= C(\, G_1{}^C) \cup C^*(\, G_2{}^C) \\ &= G_1{}^C \cup G_2{}^C = (\, G_1 \cap G_2)^C \\ &\Rightarrow (\, G_1 \cap G_2)^C \in F \\ &\Rightarrow G_1 \cap G_2 \in \mathbf{T}. \end{aligned}$$

Because all the axioms for a topology are satisfied hence $\mathbf{T}$ is a topology. We will now prove that $C^* = C$.

We have discussed above that $\mathbf{T}$ is a topology for X. Thus members of $\mathbf{T}$ are open sets therefore the closed sets are just the members of the family F.

By (K_3), $C^*[\, C^*(E)] = C^*(E)$

This implies that $C^*(E) \in F$. Now by axiom ($\mathbf{K}_2$) $E \subseteq C^*(E)$. Thus $C^*(E)$ is a closed set containing E and hence $C^*(E) \supseteq C(E)$ ...(3)

The consder that C^* (E) is the smallest closed set containing E. On the other hand by axiom ($\mathbf{K}_2$),

$E \subseteq C(E) \in F$.

So, $C^*(E) \subseteq C^*(C(E)) = C(E)$ ...(4)

Thus by Equations (3) and (4),

$C^*(E) = C(E)$ for any subset $E \subseteq X$

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6.2.3 Dense Subsets

Definition: Let A be a subset of the topological space (X, T). Then A is said to be dense in X if $\overline{A} = X$.

Trivially the entire set X is always dense in itself. $\mathbf{Q}$ is dense in $\mathbf{R}$ since $\overline{\mathbf{Q}} = \mathbf{R}$.

Let $\mathbf{T}$ be finite complement topology on $\mathbf{R}$. Then every infinite subset is dense in $\mathbf{R}$.

Theorem 5: A subset A of topological space $(X, \mathbf{T})$ is dense in X iff for every nonempty open subset B of X, $A \cap B \neq \phi$.

Proof: Suppose A is dense in X and B is a nonempty open set in X. If $A \cap B = \phi$, then $A \subseteq X–B$ implies that $\overline{A} \subset X–B$ since $X–B$ is closed. But then $X–B \underset{\neq}{\subset} X$ contradicting such that $\overline{A} = X$.

Because $\overline{A} \subset X–B \underset{\neq}{\subset} X$.

Conversely assume that A meets every nonempty open subset of X. Thus the only closed set containing A in X and consequently $\overline{A} = X$. Hence A is dense in X.

Theorem 6: In a topological space $(X, \mathbf{T})$

(i) Any set C, containing a dense set $\mathbf{D}$, is a dense set.

(ii) If A is a dense set and B is dense on A, then B is also a dense set.

Proof: (i) Since $\mathbf{D} \subset C \Rightarrow \overline{\mathbf{D}} \subset \overline{C}$

But $\overline{\mathbf{D}} = X$, hence $X \subset \overline{C}$ also $\overline{C} \subset X$ so that $\overline{C} = X$.

Thus C is dense in $(X, \mathbf{T})$.

(ii) Since A is dense, $\overline{A} = X$

Also B is dense on A.

$\Rightarrow A \subset \overline{B} \Rightarrow \overline{A} \subset \overline{\overline{B}} = \overline{B}$ (By closure property)

$\Rightarrow \overline{A} \subset \overline{B}$

$\Rightarrow X = \overline{A} \subset \overline{B}$

Thus B is dense in $(X, \mathbf{T})$.

Check Your Progress

1. What is a subbase?
2. What is a base for the topology?
3. What do you mean by an open covering?
4. What is a subcover?
5. What is a topological groups?

6.3 CLOSED SETS AND LIMIT POINTS

Let X be a metric space. All points and sets mentioned here are understood to be elements and subsets of X.

(i) If $r > 0$, the set $N(p, r) = \{x \in X: d(p, x) < r\}$

is called a neighbourhood of a point p. The number r is called the radius of $N(p, r)$.

In the metric space $\mathbf{R}, \in N(p, r) = \{x \in \mathbf{R} : |x - p| < r)$

$= \{x \in \mathbf{R} : p - r < x < p + r\} =]p - r, p + r[$

Thus in this case, neighbourhood of p is an open interval with p as a mid-point.

(ii) A point p is said to be a limit point of the set A if every neighbourhood of p contains points of A other than p.

The set of all limit points of A is called the derived set of A and shall be denoted by $\mathbf{D}(A)$.

The subset $A = \{1, \frac{1}{2}, \frac{1}{2}...\}$ of $\mathbf{R}$ has 0 as a limit point since for each $r > 0$, we can choose a positive integer n_0 such that $1/n_0 < r$ and since $1/n_0 \neq 0$, we see that every neighbourhood $N(0, r)$ of 0 contains a point of A other than 0.

The set 1 of integers has no limit point whereas the set of limit points of $\mathbf{Q}$ (the set of rationals) is all of $\mathbf{R}$ as the reader can easily verify.

(iii) A point p is called isolated point of a set A if $p \in A$ but p is not a limit point of A.

(iv) Set A is said to be closed if $\mathbf{D}(A) \subset A$, that is, if A contains all its limit points.

(v) A point p is called an interior point of A if there exists a neighbourhood N of p such that $N \subset A$.

The set of all interior points of A is called the interior of A and shall be denoted by $A°$.

For example, if $A = [0, 1]$, then $A° =]0, 1[$.

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(vi) A set A is said to be open if it contains a neighbourhood of each of its points, that is, if to each $p \in A$, there exists a neighbourhood $N(p)$ of p such that $N(p) \subset A$.

Thus A is open if and only if every point of A is an interior point of A.

Thus for every $r > 0$, the open interval $]\,p - r, p + r[$ is a neighbourhood of a point $p \in \mathbf{R}$ so a subset A of $\mathbf{R}$ is open if and only if to each $p \in A,$ there exists $r > 0$ such that

$]p - r, p + r[\subset A.$

In particular, every open interval $]a, b[$ is an open set.

For if $p \in\ ]a, b[$, take

$r = \min \{p - a, b - p\}$. Then $]p - r, p + r [\subset] a, b[$ showing that $]a, b[$ is open.

(vii) A set A is said to be perfect if A is closed and if every point of A is a limit point of A.

(viii) The closure of a set A is the union of A with its derived set $\mathbf{D}(A)$ and shall be denoted by $\overline{A}$.

(ix) A set A is said to be dense in another set B if $\overline{A} \supset B$.

Also, A is said to be dense in X or everywhere dense if $\overline{A} = X$.

(x) A set A is said to be nowhere dense or non-dense if $\overline{A}$ contains no neighbourhoods.

It is easy to see that a set A is nowhere dense if and only if $(\overline{A})^{o} = \emptyset$.

(xi) A set A is said to be bounded if there is a real number M and a point $q \in X$ such that $d(p, q) < M$ for all $p \in A$.

Theorem 7: In a metric space, every neighbourhood is an open set.

Proof: Let $N(a, r)$ be a neighbourhood of any point $a \in X$ so that $N(a, r) = \{x \in X : d(a, x) < r\}$.

If $p \in N(a, r)$ be arbitrary, then $d(a, p) < r$.

Let $\delta = r - d\,(a, p) > 0$. We shall show that,

$N\,(p, \delta) \subset N(a, r)$.

Indeed $y \in N\,(p, \delta)$ implies $d\,(p, y) < \delta$ and the triangle inequality shows that,

$$d(a, y) \leq d(a, p) + d(p, y) < d(a, p) + \delta$$
$$= d(a, p) + r - d(a, p) = r.$$

This implies that $y \in N(a, r)$.

Thus we have shown that,

$y \in N(p, \delta) \Rightarrow y \in N(a, r)$ and so $N(p, \delta) \subset N(a, r)$. Hence, $N(a, r)$ is neighbourhood of p and since p was any point in $N(a, r)$, we conclude that $N(a, r)$ is an open set.

Theorem 8: Let p be a limit point of a subset A of a metric space. Then every neighbourhood of p contains infinitely many points of A.

Proof: Suppose p has a neighbourhood N which contains only a finite number of points of A. Let $q_1,...,q_n$ be those points of $N \cap A$, which are distinct from p. Let,

$r = \min \{d(p, q) : 1 \leq i \leq n\}$.

Then $r > 0$ being the minimum of a finite set of positive numbers. The neighbourhood $N(p, r)$ contains no point of A different from p so that p is not a limit point of A which is contradiction. Hence, every neighbourhood of p must contain infinitely many points of A, thus establishing the theorem.

Corollary: A finite point set has no limit points.

Theorem 9: A set A is open if and only if its complement is closed.

Proof: Suppose A is open. To show that its complement A′ is closed. Let x be any limit point of A′. Then every neighbourhood of x contains a point of A'. This implies that no neighbourhood of x can be contained in A and so x is not an interior point of A. Since A is open, this means that $x \in A'$ and consequently A' is closed.

Conversely, let A' be closed and let x be an arbitrary point of A. Then $x \notin A'$. Since A' is closed x cannot be limit point of A'. Hence, there exists a neighbourhood N of x such that N contains no point of A', that is, $N \subset A$. Thus, A contains a neighbourhood of each of its points and so A is open.

Theorem 10: Let A be a closed subset of **R** which is bounded above. If u be the least upper bound or lub of A, then $u \in A$.

Proof: Suppose $u \notin A$. For every $h > 0$, there is a point $x \in A$ such that $u - h < x \leq u$, for otherwise $u - h$ would be an upper bound of A. Thus every neighbourhood of u contains a point x of A. Also $x \neq u$ since $u \notin A$. It follows that u is a limit point of A, which is not a point of A. Therefore, A is not closed which contradicts the hypothesis. Hence $u \in A$ as desired.

Note: If A is closed and bounded below and if l is greatest lower bound or glb of A, then $l \in A$.

Proof is similar to that of the preceding theorem.

Theorem 11: Let (X, d) be a metric space. Then,

(i) The empty set $\varnothing$ and the whole space X are open as well as closed.

(ii) Union of an arbitrary collection of open sets is open.

(iii) Union of finite number of closed sets is open.

(iv) Intersection of an arbitrary collection of closed sets is closed.

Proof: To prove case (i), let $\{A_\lambda : \lambda \in A\}$ be an arbitrary collection of open sets.

Let, $\quad A = \cup \{A_\lambda : \lambda \in A\}$.

$x \in A \Rightarrow x \in A$ for some $\lambda \in A$

$\Rightarrow$ There exists $\in > 0$ such that $N(x, \in) \subset A_\lambda \quad [\because A\lambda \text{ is open}]$

$\Rightarrow N(x, \in) \subset A \in \quad [\because A\lambda \subset A]$

$\Rightarrow A$ is open (By definition).

Proofs of Cases (ii) and (iii) at once follow by using De Morgan laws for complements. Thus to prove case (ii) let A_i, $i = 1, 2, ..., n$ be a finite collection of closed sets. Then,

A_i is closed $\quad \Rightarrow X - A_i$ is open $\forall\, i = 1, 2, ..., n$.

$\Rightarrow \cap \{X - A_i : i = 1, 2, ..., n\}$ is open by case (iii).

$\Rightarrow X - \cup \{Ai : i = 1, 2, ..., n\}$ is open.

[By De Morgan Law]

$\Rightarrow \{A_i : i = 1, 2, ..., n\}$ is closed.

6.3.1 Neighbourhoods

A subset N of **R** is called a neighbourhood of a point $p \in$ **R** if N contains an open interval containing p and contained in N, that is, if there exists on open interval $]a, b[$ such that,

$$p \in]a, b[\subset N.$$

It immediately follows that an open interval is a neighbourhood of each of its points. For practical purposes, it will therefore suffice to take open intervals containing a point of its neighbourhoods.

Note if]a, b[is an open interval containing a point p so that $a < p < b$, we can always find an $\varepsilon > 0$, such that $]p - \varepsilon, p + \varepsilon[\subset]a, b[$. Choose any ε less than the smaller of the two numbers $p–a$ and $b–p$]. Clearly $]p–\varepsilon, p + \varepsilon[$ is an open interval containing p and so it is a neighbourhood of p. We shall use this form of neighbourhood of p, usually called an e-neighbourhood of p and shall denote it by

$N(p, \varepsilon)$. We call ε as the radius of $N(p, \varepsilon)$. The point p itself is a mid-point of centre of $N(p, \varepsilon)$. It is evident that,

$$x \in , (p, \varepsilon) \text{ if } |x - p| < \varepsilon,$$

We shall use the abbreviated form '*nhd*' for the word 'neighbourhood'.

(i) The closed interval [1, 3] is an *nhd* of 2 since $]\frac{3}{2}, \frac{5}{2}[$ is an open interval such that $2 \in][\frac{3}{2}, \frac{5}{2}[\subset [1, 3]$. But [1, 3] is not an *nhd* of 1 since there exists no open interval which contains 1 and contained in [1, 3]. Similarly [1, 3] is not an *nhd* of 3.

(ii) The set N of natural numbers is not *nhd* of any of its points since no open interval can be a subset of N.

6.3.2 Open Sets

A subset G of **R** is called open if for every point $p \in G$, there exists an open interval I such that $p \in I \subset G$.

This is equivalent to saying that G is open if for every $p \in G$, there exists ε *nhd* $N(p,\varepsilon) =]p–\varepsilon, p + \varepsilon[$such that $N(p, \varepsilon) \subset G$.

For example,

(i) Every open interval is an open set.

(ii) The empty set ϕ and the whole real line **R** are open sets. Since ϕ contains no points, the preceding definition is satisfied. Hence ϕ is open.

To show that **R** is open, we observe that for every $p \in$ **R** and every $\varepsilon > 0$, we have $]p–\varepsilon, p + \varepsilon[\subset \mathbf{R}$. Hence, **R** is open.

Now consider the following examples,

(i) The closed open interval [2, 3[is not open, since there exists no ε-*nhd* of 2 contained in [2, 3[.

(ii) The set $A=\{1/n : n \in N\}$ is not open since no point of A has an ε-*nhd* contained in A.

Theorem 12: The union of any collection of open sets is an open set.

Proof: Let C be any collection of open sets and let S be their union, that is, let $S = \cup \{G : G \in C\}$. Let $p \in S$. Then p must belong to at least one of the sets in C, say $p \in G$. Since G is open there exists and ε-*nhd* $N(p, \varepsilon)$ of p such that $N(p, \varepsilon) \subset G$. But $G \subset S$, and so $N(p, \varepsilon) \subset S$. Hence S is open.

Theorem 13: The intersection of a finite collection of open sets is open.

Proof: Let $S = \bigcap_{i=1}^{n} G_i$, where each G_i is open. Assume $p \in S$.

If S is empty, there is nothing to prove. Then $p \in G_i$ for every i=1, 2,....,n. Since G_i is open, there exists $\varepsilon_i > 0$ such that $N(p, \varepsilon_i) \subset G_i$, for every i = 1, 2,..., n. Let ε = min [ε_1, ε_2,...., ε_n].

Then $N(p, \varepsilon) \subset \bigcap_{i=1}^{n} G_i = S$. Hence S is open.

Note: The intersection of an infinite collection of open sets is not necessarily open. For example, if G_n =]–1/n, 1/n] ($n \in N$), then each G_n is open (being an open interval), but $\bigcap_{i=1}^{n} G_n = \{0\}$ which is not open since there exists no $\varepsilon > 0$ such that]– ε, ε[$\subset \{0\}$.

Example 1: Show that complement of every singleton set in **R** is open. More generally, the complement of a finite set is open.

Solution: Let $\{x\}$ be a singleton set in **R**. To show that its complement $\{x\}$' is open. Let $y \in \{x\}$'. If $\{x\}$' = ϕ, there is nothing to prove. Then $y \neq x$. Set $|x - y| = r > 0$. Let $0 < \varepsilon < r$. Then $N(y, \varepsilon)$=] y–ε, $y + \varepsilon$[does not contain x and therefore $N(y, \varepsilon) \subset \{x\}$'. Hence $\{x\}$' is open. Again, if A=$\{x_1, x_2..., x_n\}$ is any finite subset of **R**, then we can write **A**=$\{x_1\} \cup \{x_2\} \cup ... \cup \{x_n\}$. Then A'=[$\{x_1\} \cup ... \cup \{x_n\}$]'= $\{x_1\}' \cap ... \cap \{x_n\}'$. Since each $\{x_1\}$' is open.

6.3.3 Closed Sets

Definition: A set F in **R** is called closed if its complement F is open.

For example, (i) Every closed interval $[a,b]$ is closed, since its complement $[a,b]'$ =] $-\infty$, a[$\cup$] b, ∞[is open, being a union of two open intervals.

(ii) Every singletion set in **R** is closed. More generally every finite set in **R** is closed.

(iii) The closed open interval [a,b[is closed.

Theorem 14: (i) The union of a finite collection of closed sets is closed.

(ii) The intersection of an arbitary collection of closed sets is closed.

Proof: (i) Let F_i (i = 1,2,..., n) be a finite collection of closed sets. Then each F_i' is open. By De Morgan law, we have F_i' which is open, being the intersection of finite collection of open sets. Hence F_i is closed.

Let C be an arbitrary collection of closed sets. Then each $F \in C$ is closed and so its complement F' is open. By De Morgan law, we have $(\bigcap F)'=(F \in C)$, which is open. Hence $\bigcap F\,(F \in C)$ is closed.

For example if A is open and B is closed, then show that AB is open sets and $BA=B \bigcap A'$, which is the intersection of two closes sets. Hence, AB is open and BA is closed.

6.3.4 Accumulation Points or Limit Points: Adherent points

If $A \subset \mathbf{R}$, then a point $p \in \mathbf{R}$ is called an accumulation point (or a limit point) of A if every ε-*nhd* $N(p, \varepsilon)$ of p contains a point of A distinct from p.

The set of all accumulation points of A is called the first derived set (or simply the derived set) of A and is denoted by $\mathbf{D}(A)$. The first derived set of $\mathbf{D}(A)$ is called the second derived set of A and is denoted by $\mathbf{D}^2(A)$.In general, nth derived set of A is denoted by $\mathbf{D}^n(A)$.

For example,

(i) Every point of the closed interval $[a,b]$ is an accumulation point of the set of points in the open interval $]a,[b$. So in this case $\mathbf{D}]a,b[=a,b]$.

(ii) 0 is the only accumulation point of the set $A=\{1/n{:}n \in N\}$.

Hence $\mathbf{D}(A)=\{0\}$.

(iii) Every real number is an accumulation point of the set Q of rational numbers and so $\mathbf{D}(\mathbf{Q}) = \mathbf{R}$.

(iv) If $A = [2,3[$, then $\mathbf{D}\,(A) = [2,3]$.

If $A \subset \mathbf{R}$, then a point $p \in \mathbf{R}$ is called an adherent point of A if every ε-*nhd* of p contains a point of A. The set of all adherent points of A is called the adherence of A denoted by $Adh\,(A)$.

Theroem 15: If p is an accumulation point of A,then every $\in$ *nhd* of p contains infinitely many points of A.

Proof: Assume the contrary, that is suppose there exists an $\in$ *nhd* $(b, \in) = p + \in [$of p which cantains only a finite number of point distinct from p, say $p_1, p_2, \ldots p_n$. Let r denote the smallest of the positive numbers.

$$|p-p_1|,\ |p-p_a|\ , \ldots,\ |p-p_n|.$$

Then $N(p, r/2) =]p-r/2, p+r/2[$ is an ε-*nhd* of p which contains no points of A distinct from p. This is a contradiction. Hence every ε-*nhd* of p must contain infinitely many points of A.

A set is said to be of **first species** if it has only a finite number of derived sets. It is said to be of **second species** if the number of its derived sets is infinite.

Note that if a set is of first species, then its last derived set must be empty.

A set whose $(n + 1)^{th}$ derived set is empty is called a set of ***n*th order**.

For example, the set **Q** of all rational numbers is of second species, since

$$\mathbf{D}(\mathbf{Q}) = \mathbf{R},\ \mathbf{D}^2(\mathbf{Q}) = \mathbf{D}(\mathbf{R}) = \mathbf{R}.$$

$$\mathbf{D}^3(\mathbf{Q}) = \mathbf{D}(\mathbf{R}) = \mathbf{R}, \text{ etc.}$$

Hence, all the derived sets of **Q** are equal to **R**.

Consider that $A = \{1/n : n \in N\}$. Then $\mathbf{D}(A) = \{0\}$. Since $\mathbf{D}(A)$ consists of a single point, it cannot have any limit point and so $\mathbf{D}^2(A) = \phi$. Therefore A is of first species and first order.

Again let $A = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{2}+\frac{1}{3}, \frac{1}{4}, \frac{1}{2}+\frac{1}{4}, \frac{1}{5}, \frac{1}{2}+\frac{1}{5} ....\}$. Then A has the limit point 0 and also the limit point $\frac{1}{2}$, for the subset $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...\}$ is dense at the origin and the subset $\{\frac{1}{2}, \frac{1}{2}+\frac{1}{3}, \frac{1}{2}+\frac{1}{4}, ...\}$ is dense at $\frac{1}{2}$. Therefore $\mathbf{D}(A) = \{0, \frac{1}{2}\}$ and $\mathbf{D}^2(A) = \phi$. Hence A is of first species and first order.

For example, let $A = \{1, \frac{1}{2}, (\frac{1}{2})^2, \frac{1}{2} + (\frac{1}{2})^2, (\frac{1}{2})^3, \frac{1}{2} + (\frac{1}{2})^3, (\frac{1}{2})^2 + (\frac{1}{2})^3, (\frac{1}{2})^4, \frac{1}{2} + (\frac{1}{2})^4, (\frac{1}{2})^2 + (\frac{1}{2})^4, (\frac{1}{2})^3 + (\frac{1}{2})^4, (\frac{1}{2})^5, ...\}$.

Then it is clear by inspection that,

$$\mathbf{D}(A) = \{0, \frac{1}{2}, (\frac{1}{2})^2, (\frac{1}{2})^3, ...\},\ \mathbf{D}^2(A) = \{0\} \text{ and } \mathbf{D}^3(A) = \phi.$$

Hence in this case, A is of first species and second order.

Let the points of a set A be given by,

$$\frac{1}{3^{S_1}} + \frac{1}{5^{S_2}} + \frac{1}{7^{S_3}} + \frac{1}{11^{S_4}},$$

Where S_1, S_2, S_3, S_4 each have all positive integral values.

Hence $\mathbf{D}(A)$ consists of the four sets of points given by,

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$$\frac{1}{3^{S_1}}+\frac{1}{5^{S_2}}+\frac{1}{7^{S_3}},\ \frac{1}{3^{S_1}}+\frac{1}{5^{S_2}}+,\frac{1}{11^{S_4}},\ \frac{1}{3^{S_1}}+\frac{1}{7^{S_3}},\frac{1}{11^{S_4}}$$

$\frac{1}{5^{S_2}}+\frac{1}{7^{S_3}}+\frac{1}{11^{S_4}}$ and the six sets of the points,

$$\frac{1}{3^{S_1}}+\frac{1}{5^{S_2}},\ \frac{1}{3^{S_1}}+\frac{1}{7^{S_2}},\ \frac{1}{3^{S_1}}+\frac{1}{11^{S_4}},\ \frac{1}{5^{S_2}}+\frac{1}{7^{S_3}}$$

$\frac{1}{5^{S_2}}+\frac{1}{11^{S_4}},\ \frac{1}{7^{S_3}}+\frac{1}{11^{S_4}}$ and four sets of the points,

$\frac{1}{3^{S_1}},\frac{1}{5^{S_2}},\frac{1}{7^{S_3}},\frac{1}{11^{S_4}}$ together with the single point 0.

$\mathbf{D}^2(A)$ consists of the last ten of these sets and the point 0, $\mathbf{D}^3(A)$ consists of the last four sets and the point 0 and $\mathbf{D}^4(A)$ consists of the point 0 only. Thus, the set A is of the first species and fourth order.

Example 2: Show that, zeros of sin $(1/x)$ form a set of first order, zeros of sin $\left(\frac{1}{\sin\frac{1}{x}}\right)$ form a set of second order; zeros of sin $\left(\frac{1}{\sin\frac{1}{\sin\frac{1}{x}}}\right)$ form a set of third order, and so on.

Solution: The zeros of $\sin\frac{1}{x}$ are given by $\sin\frac{1}{x}=0$, which gives $\frac{1}{x}=n\pi$ or $x=\frac{1}{n\pi}$, where n is an integer. Hence if A is the set of zeros of $\sin\frac{1}{x}$, then A consists of all of the form $\frac{1}{n\pi}$ where n is an integer. Clearly $\mathbf{D}(A) = \{0\}$ and so A is of first order.

Again $\sin\dfrac{1}{\sin\frac{1}{x}}=0$, gives $\dfrac{1}{\sin\frac{1}{x}}=n\pi$, where n is an integer.

Then $\sin\frac{1}{x} = \frac{1}{n\pi}$, from which the general value $\frac{1}{x}$ is given by,

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$$\frac{1}{x} = n\pi + \sin^{-1} = \frac{1}{n\pi} \text{ where } m \text{ is an integer}$$

or $$x = \frac{1}{m\pi + \sin^{-1}\left(\frac{1}{n\pi}\right)} \quad \text{....(1)}$$

Thus the zeros of $\sin\left(\frac{1}{\sin\frac{1}{x}}\right)$ form a set, say B, of points of the form (1). It is clear that $\mathbf{D}(B)$ consists of points of the form $\frac{1}{m\pi}$ and the point 0 and so $\mathbf{D}^2(B)$ consists of the single point 0. Hence B is of second order.

Similarly we can show that the zeros of $\sin\left(\frac{1}{\sin\frac{1}{\sin\frac{1}{x}}}\right)$ form a set of third order, and so on.

Theorem 16 (Bolzano-Weierstrass Theorem): A bounded infinite set of real numbers has at least one limit point.

Proof: Let A be an infinite bounded subset of $\mathbf{R}$. Then, there exist finite constants m and M such that $m \le a \le$ M for all $a \in A$. At least one of the intervals [m, (m+M), M] must contain an infinite number of points of A. We rename such an interval as $[a_1+b_1]$. Similarly, one of the intervals $[a_1, \frac{1}{2}(a_1+b_1), [a_1+b_1], [\frac{1}{2}(a_1+b_1), b_1]$ contains infinitely many points of A and we desingnate it as $[a_2,b_2]$. Now proceed with $[a_2, b_2]$ as we did with $[a_1, b_1]$. Continuing in this way, we obtain a sequence of closed intervals $\{I_n\}=$

$\{[a_n, b_n]\}$ such that $I_n \supset I_{n+1}$ and $[I_n] = b_n - a_n = \dfrac{M-m}{2^n} \to 0$ as $n \to \infty$. Hence I_n consists of a single point, say x_0. We choose n such that $b_n - a_n < \varepsilon$. Then $I_n \subset]x_o - \varepsilon, x_0 + \varepsilon[$ and consequently the interval $]x_o - \varepsilon, x_0 + \varepsilon[$contains an infinite number of points of A. It follows that x_0 is a limit point of A. This completes the proof.

6.3.5 Closed Sets and Accumulation Points

A closed set was defined to be the complement of an open set. The Theorem 1.17 describes closed sets in another way.

Theorem 17: A set A is closed if and only if it contains all its accumulation points**.**

Proof: Assume A closed and P is an accumulation point of A. We have to prove that $P \in A$. suppose, if possible, $P \notin A$. Then $P \in A,$ and since $A,$ is an open, there exists an $\in$-*nhd* $N(p, \in) =]sP-, P+\in[$such that $N(p, \in) \subset A$. Consequently $N(p, \in)$ contains no point of A, contradicting the fact that p is an accumulation point of A. Hence, we must have $P \in A$.

Conversely, we assume that A contains all its accumulation points and show that A is closed. Let $P \in A'$. Then $P \notin A$, so by hypothesis p is not an accumulation point of A. Hence there exists an $\in$-*nhd* $N(p, \in)$ of p which contains no point of A other then p. Since $P \notin A$, $N((p, \in)$ contains no point of A and consequently N $(p, \in) \subset \forall A'$. Therefore A' is open, and hence A is closed.

Theorem 18: All the derived sets $\mathbf{D}(A)$, $\mathbf{D}^2(A)$, ... $\mathbf{D}^n(A)$,... of a given set A are closed sets and each of these derivatives, after the first, consists of points belonging to the preceding one and therefore to $\mathbf{D}(A)$.

Proof: We first prove that $\mathbf{D}(A)$ is closed. Let p be any accumulation point (limit point) of $\mathbf{D}$. Then every $\in$-*nhd* of p contains infinitely many points of $\mathbf{D}(A)$ and since point of $\mathbf{D}(A)$ is an accumulation point of A, every $\in$-*nhd* of p must contain infinitely many points of A. Thus, p is also a limit point of A and so $p \in \mathbf{D}(A)$. Therefore, $\mathbf{D}(A)$ contains all its accumulation points and so $\mathbf{D}(A)$ is closed. Again since $\mathbf{D}^2(A)$ is the derived set of $\mathbf{D}(A)$, and so it must be closed as proved earlier. Similarly $\mathbf{D}^2(A)$, ..$\mathbf{D}^n(A)$, ... are closed sets.

To prove the second part of the theorem, let $x \in \mathbf{D}^n(A)$, $n \geq 2$ and suppose, if possible, that $p \notin \mathbf{D}(A)$. Then there can be found an $\in$-*nhd* of x which contains only a finite number of points of A or no such points and this *nhd* would therefore contain no points of $\mathbf{D}(A)$ and consequently it can contain no point of $\mathbf{D}^2(A)$. $\mathbf{D}^3(A)$, ..., $\mathbf{D}^n(A)$ which is contrary to the hypothesis that $x \in \mathbf{D}^n(A)$. Hence every point of $\mathbf{D}^n(A)$ $(n \geq 2)$ must belong to $\mathbf{D}(A)$.

6.3.6 Closure

The closure of a set A in **R** is the smallest closed set containing A and is denoted by $\overline{A}$.

By this definition , $A \subset \overline{A}$. Also $\overline{A}$ is always a closed set of A.

Theorem 19: Let A be a set in **R**. Then $\overline{A} = \mathrm{A} \cup \mathrm{D}\,(A)$, i.e., A is the set of all adherent points of A.

Proof: We first observe that $A \cup \mathbf{D}\,(A)$ is closed. For if p is any limit of $A \cup \mathbf{D}\,(A)$. then either p is a limit point of A or a limit of $\mathbf{D}\,(A)$. If p is a limit point of A, then $p \in \mathbf{D}\,(A)$. If p is a limit point of $\mathbf{D}(A)$, then since $\mathbf{D}\,(A)$ is closed $p \in \mathbf{D}\,(A)$. So on either case, $p \in \mathbf{D}\,(A)$ and surely then $p \in A \cup \mathbf{D}\,(A)$. Hence $A \cup \mathbf{D}\,(A)$ is closed.

Now $A \cup \mathbf{D}\,(A)$ is a closed set containing A, and since $\overline{A}$ is the smallest closed set containing A, we have

$$\overline{A} \subset A \cup \mathbf{D}\,(A). \qquad \text{...(5)}$$

Also $A \subset \overline{A} \Rightarrow \mathbf{D}\,(A) \subset \mathbf{D}(A)$ .. (6)

Since $\overline{A}$ is closed, we have $\mathbf{D}\,(\overline{A}) \subset \overline{A}$. ... (7)

$\therefore$ From Equations (6) and (7), $\mathbf{D}\,(A) \subset \overline{A}$.

Moreover, $A \subset \overline{A}$, and $\mathbf{D}\,(A)\, A \subset \overline{A} \Rightarrow A \cup \mathbf{D}\,(A) \subset \overline{A}$ .... (8)

$\therefore$ From Equations (5) and (8), $\overline{A} = A \cup \mathbf{D}\,(\boldsymbol{A})$.

Theorem 20: Let A, B be subsets of **R**, Then:

(i) $A \subset B \Rightarrow \overline{A} \subset \overline{B}$, (ii) $\overline{A \cup B} \subset \overline{A} \cup \overline{B}$.

(iii) $\overline{A \cup B} \subset \overline{A} \cup \overline{B}$, (iv) $\overline{\overline{A}} = \overline{A}$.

Proof: (i) Given $A \subset B$. But $B \subset \overline{B}$ always. Hence $A \subset B$, Thus $\overline{B}$ is a closed set containing A. But $\overline{A}$ is the smallest closed set containing A. Hence $\overline{A} \subset \overline{B}$.

(ii) $A \subset A \cup B \Rightarrow \overline{A} \subset \overline{A \cup B}$ and $B \subset A \cup B \Rightarrow \overline{B}\ \overline{A \cup B}$ by case (i).

Hence, $\overline{A} \cup \overline{B} \subset \overline{A \cup B}$. ... (9)

Again $A \subset \overline{A}$ and $B \subset \overline{B} \Rightarrow \mathrm{A} \cup B \subset \overline{A} \cup \overline{B}$. But $\overline{A} \cup \overline{B}$ is closed, being the union of two closed sets. Thus $\overline{A} \cup \overline{B}$ is a closed set containing $A \cup B$. But $\overline{A \cup B}$ is the smallest closed set containing $A \cup B$. Hence $\overline{A \cup B} \subset \overline{A} \cup \overline{B}$

.... (10)

$\therefore$ From Equations (9) and (10), $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

(iii) $A \cap B \subset A \Rightarrow \overline{A \cap B} \subset \overline{A}$ and $A \cap B \subset B \Rightarrow A \cap B \subset \overline{B}$ by case (*i*).

Hence $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$.

(iv) Since $\overline{A}$ is closed, hence $\overline{\overline{A}} = \overline{A}$.

Consider the following examples:

(i) If **Q** is the set of rational numbers, then

$\mathbf{D}(\mathbf{Q}) = \mathbf{R}$ and so $\overline{Q} = \mathbf{Q} \cup \mathbf{D}(\mathbf{Q}) = \mathbf{Q} \cup \mathbf{R} = \mathbf{R}$.

(ii) Let $A =]2, 3[$. Then $\mathbf{D}(A) = [2, 3]$ and so

$\overline{A} = A \cup \mathbf{D}(A) = [2, 3]$.

(iii) Let A be any finite set in **R**. Since every finite set of real numbers is closed, we have $\overline{A} = A$.

6.4 HAUSDORFF SPACES

Theorem 21: Let X be a Hausdorff topological space. Then no net in X can have two different limits.

Proof: Let X be a Hausdorff topological space and $(x_\lambda)_{\lambda \in \wedge}$ be a net in X. Suppose $a, b \in X$ and we have both $x_\lambda \to a$ and $x_\lambda \to b$. We claim $a = b$ must hold.

If not, by the Hausdorff property of X, there are U, $V \subset X$ open, disjoint with $a \in U$ and $b \in V$. By the definition of $x_n \to a$, there is $\lambda_a \in \Lambda$ so that $x_\lambda \in U \ \forall \lambda \geq \lambda_a$. By the definition of $x_\lambda \to b$, there is $\lambda_b \in \Lambda$ so that $x_\lambda \in B \ \forall \lambda \geq \lambda_b$. Since Λ is directed, there is $\lambda \in \Lambda$ with $\lambda \geq \lambda_a$ and $\lambda \geq \lambda_b$. Then both $x_\lambda \in U$ and $x_\lambda \in V$ hold. But this leads to the contradiction $x_\lambda \in U \cap V = \theta$.

Theorem 22: Let X be a topological space in which there is no net with two different limits. Then X is Hausdorff.

Proof: Take $a, b \in X$ with $a \neq b$. Our claim is that there exist U and V open with $a \in U$, $b \in V$ and $U \cap V \neq \phi$. If not, we have $U \cap V \neq \phi$ for any open $U, V \subseteq X$ where $a \in U$, $b \in V$.

From this we need to find a net with two limits (which will be a and b). We take for the index set $\Lambda = \{(U, V) : U, V \subseteq X \text{ open, } a \in U, b \in V\}$ where we define the order $(U_1, V_1) \leq (U_2, V_2)$ to mean that both $U_1 \supseteq U_2$ and $V_1 \supseteq V_2$ hold. It is straightforward to check that Λ is then a directed set. The least trivial part is to show that $(U_1, V_1), (U_2, V_2) \in \Lambda$ implies there is $(U_3, V_3) \in \Lambda$ with $(U_1, V_1) \leq (U_3, V_3)$

and $(U_2, V_2) \leq (U_3, V_3)$. We just take $U_3 = U_1 \cap U_2$ and $V_3 = V_1 \cap V_2$ and check U_3 and V_3 open, $a \in U_3$, $b \in V_3$ and the four containments we need.

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For each $\lambda = (U, V) \in \Lambda$ choose one $x_\lambda \in \mathrm{U} \cap \mathrm{V}$. Then $(x_\lambda)_{\lambda \in \Lambda}$ is a net in X and we claim both $x_\lambda \to a$ and $x_\lambda \to b$.

To verify $x_\lambda \to a$, take a neighbourhood $N_a \in \mathcal{U}_a$. Then $\lambda_0 = ((N_a)^\circ, X) \in \Lambda$. If $\lambda = (U, V) \in \Lambda$ satisfies $\lambda \geq \lambda_0$, then $x_\lambda \in U \cap V \subseteq U \subseteq (N_a)^\circ \subseteq N_a$. To verify $x_\lambda \to b$ we argue in a similar way but take $\lambda_0 = (X, (N_b)^\circ)$.

Check Your Progress

6. What is a neighbourhood of a point?
7. What is an open set?
8. What is an accumulation point?

6.5 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1. Let (X, T) be a topological space. A set $S \subset \mathbf{T}$ is called a subbase if $\{S_1 \cap ..... \cap S_n$: $S_j \in S$, $n \in \mathbf{Z}\}$ is a base for $\mathbf{T}$. Equivalently, $\mathbf{T} = \bigcup \{B \subset X : \mathcal{B} \bigcap \mathcal{A}$ for some finite subset $\mathcal{A} \subset S\}$.
2. A system $\mathcal{B}$ of subsets of X is called a basis (or base) for the topology $\mathcal{U}$ if every open set is the union of certain sets in $\mathcal{B}$. Equivalently, for each open set U, given any point $x \in U$, there exists $B \in \mathcal{U}$ such that $x \in B \subset U$.
3. An open covering of a topological space X is a family of open sets having the property that every $x \in X$ is contained in at least one set in the family.
4. A subcover of an open covering is an open covering of X which consists of sets belonging to the open covering.
5. A topological group G is a group that is also a topological space, having the property the maps $(g_1, g_2) \to g_1 g_2$ from $G \times G \to G$ and $g \to g^{-1}$ from G to G are continuous maps.
6. A subset N of $\mathbf{R}$ is called a neighbourhood of a point $p \in \mathbf{R}$ if N contains an open interval containing p and contained in N, that is, if there exists on open interval $]\,a, b\,[$ such that,

$$p \in \,]a, b[\, \subset N.$$

7. A subset G of **R** is called open if for every point $p \in G$, there exists an open interval I such that $p \in I \subset G$.
8. If $A \subset \mathbf{R}$, then a point $p \in \mathbf{R}$ is called an accumulation point (or a limit point) of A if every ε-*nhd* $N(p, \varepsilon)$ of p contains a point of A distinct from p.

6.6 SUMMARY

1. Any nonempty collection $\mathbf{C} \subset \mathcal{P}(X)$ defines a topology **T** such that **C** is a subbase for **T**.
2. If $\mathcal{U}$ defines a topology on X, then we say that X is a topological space.
3. If X and Y are topological spaces, then there is a natural topology on the Cartesian product $X \times Y$ that is defined in terms of the topologies on X and Y, called the product topology.
4. A function $f: X \to Y$ is a homeomorphism of X onto Y if f is bijective and both f and f^{-1} are continuous functions.
5. A subset Y of a topological space X is compact if Y is compact in the subspace topology.
6. A topological space X is Hausdorff (or $\mathbf{T}_2$) if given distinct points x and $y \in X$, there exist neighbourhoods U of x and V of y such that $U \cap V = \phi$.
7. We define a topology $\mathrm{U}_{G/H}$ on G/H, called the quotient topology, by $\mathcal{U}_{G/H} = \{p(U) \mid U \in \mathcal{U}_G\}$. Here, $\mathcal{U}_G$ is the topology on G.
8. If G and G' are topological groups, a map $f: G \to G'$ is a continuous homomorphism of G into G' if f is a homomorphism of groups and f is a continuous function.
9. The elements of the collection are called the open sets of the topology. The openness of a set is not a property of the set itself but it refers only to the membership of the set in the collection of subsets which is called the topology.
10. A set is defined as being closed with respect to a topology if its complement is open with respect to the topology; i.e., if its complement belongs to the topology.
11. In general topology, if X is a topological space and A is a subset of X, then the *closure* of A in X is defined to be the smallest closed set containing A or equivalently the intersection of all closed sets containing A.
12. The closure operator C that assigns to each subset of A its closure $C(A)$ is thus a function from the power set of X to itself. The closure operator satisfies the following axioms:

- Isotonicity: Every set is contained in its closure.
- Idempotence: The closure of the closure of a set is equal to the closure of that set.
- Preservation of Binary Unions: The closure of the union of two sets is the union of their closures.
- Preservation of Nullary Unions: The closure of the empty set is empty.

13. Let A be a subset of the topological space (X, T). Then A is said to be dense in X if $\overline{A} = X$.
14. A point p is called isolated point of a set A if $p \in A$ but p is not a limit point of A.
15. Set A is said to be closed if $\mathbf{D}(A) \subset A$, that is, if A contains all its limit points.
16. A point p is called an interior point of A if there exists a neighbourhood N of p such that $N \subset A$.
17. A set A is said to be open if it contains a neighbourhood of each of its points, that is, if to each $p \in A$, there exists a neighbourhood $N(p)$ of p such that $N(p) \subset A$.
18. A set A is said to be perfect if A is closed and if every point of A is a limit point of A.
19. The closure of a set A is the union of A with its derived set $\mathbf{D}(A)$ and shall be denoted by $\overline{A}$.
20. The union of any collection of open sets is an open set.
21. The union of a finite collection of closed sets is closed.
22. The intersection of an arbitary collection of closed sets is closed.
23. If $A \subset \mathbf{R}$, then a point $p \in \mathbf{R}$ is called an accumulation point (or a limit point) of A if every ε-*nhd* $N(p, \varepsilon)$ of p contains a point of A distinct from p.
24. A set is said to be of first species if it has only a finite number of derived sets. It is said to be of second species if the number of its derived sets is infinite.
25. A bounded infinite set of real numbers has at least one limit point.
26. A set A is closed if and only if it contains all its accumulation points.
27. The closure of a set A in $\mathbf{R}$ is the smallest closed set containing A and is denoted by $\overline{A}$.
28. Let X be a Hausdorff topological space. Then no net in X can have two different limits.
29. Let X be a topological space in which there is no net with two different limits. Then X is Hausdorff.

6.7 KEY WORDS

- **Open covering:** An open covering of a topological space X is a family of open sets having the property that every $x \in X$ is contained in at least one set in the family.
- **Sub Cover:** A subcover of an open covering is an open covering of X which consists of sets belonging to the open covering.
- **Topological groups:** A topological group G is a group that is also a topological space, having the property the maps $(g_1, g_2) \to g_1g_2$ from $G \times G \to G$ and $g \to g^{-1}$ from G to G are continuous maps.
- **Limit point:** A point p is said to be a limit point of the set A if every neighbourhood of p contains points of A other than p.
- **Neighbourhood:** A subset N of $\mathbf{R}$ is called a neighbourhood of a point $p \in \mathbf{R}$ if N contains an open interval containing p and contained in N, that is, if there exists on open interval $]a, b[$ such that,

$$p \in]a, b[\subset N.$$

- **Open sets:** A subset G of $\mathbf{R}$ is called open if for every point $p \in G$, there exists an open interval I such that $p \in I \subset G$.

6.8 SELF ASSESSMENT QUESTIONS AND EXERCISES

Short Answer Questions

1. Give a brief description of subspace and relative topology.
2. Interpret topology in terms of Kuratowski closure operator and Neighbourhood systems.
3. Describe dense subsets briefly.
4. Write a short note on limit points.
5. Discuss Bolzano-Weierstrass theorem in brief.

Long Answer Questions

1. Discuss closed sets and limit points in detail
2. Give a detailed account of Kuratowski clouse axioms.
3. Give a comparative account of closed open sets and closed sets.
4. Explain Hausdorff spaces with suitable theorems.

6.9 FURTHER READINGS

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NOTES

UNIT 7 CONTINUOUS FUNCTION, PRODUCT TOPOLOGY AND HOMEOMORPHISM

Structure

7.0 INTRODUCTION

A function for which sufficiently small changes in the input result in arbitrarily small changes in the output is known as a continuous function. Otherwise, a function is called a discontinuous function. A continuous function with a continuous inverse function is known as a homeomorphism. Continuity of functions is one of the core concepts of topology. A product space is the Cartesian product of a family of topological spaces equipped with a natural topology known as the product topology. In this unit you will study continuous functions and continuity of a function. You will understand characterisation of continuous functions using pre-images. You will learn general properties of continuous functions. You will describe homeomorphisms. The product topology is also discussed in this unit.

7.1 OBJECTIVES

After going through this unit, you will be able to:

- Understand continuous function, continuity of functions and general properties of continuous functions
- Analyse characterisation of continuous functions using pre-images
- Describe homeomorphisms
- Explain the importance of the product topology

7.2 CONTINUOUS FUNCTIONS AND CONTINUITY OF A FUNCTION

NOTES

$D \subset R$ is compact if and only if for any given open covering of D we can subtract a finite sucovering. That is, given $(G_\alpha)_{\alpha \in A}$, a collection of open subsets of R (A an arbitrary set of indices) such that $D \subseteq \bigcup_{\alpha \in A} G_\alpha$, then there exist finitely many indices $\alpha_1, ..., \alpha_n \in A$ such that $D \subseteq \bigcup_{i=1}^{N} G_{\alpha_i}$.

Let D be an arbitrary subset of R. Then $A \subseteq D$ is open in D (or relative to D, or D-open) if and only if there exists G open subset of R such that $D = G \cap D$. Similarly we can define the notion of D-closed sets. Note that D is both open and closed in D and so is $\emptyset$.

$D \subseteq R$ is connected if and only if $\emptyset$ and D are the only subsets of D which are both open in D and closed in D. In other words, if $D = A \cup B$ and A, B are disjoint D-open subsets of D, then either $A = \emptyset$ or $B = \emptyset$

Let $D \subseteq R$, $a \in D$ a fixed element and $f : D \to R$ an arbitrary function. By definition, f is continuous at a if and only if the following property holds:

$$\forall \varepsilon > 0, \exists \delta_a(\varepsilon) > 0$$

such that $|x - a| \le \delta_a(\varepsilon)$ and $x \in D \Rightarrow |f(x) - f(a)| \le \varepsilon$

The last implication can be rewritten in terms of sets as follows:

$$f\left(B_a\left(\delta_a(\varepsilon)\right) \cap D\right) \subseteq B_{f(a)}(\varepsilon)$$

Here, we use the notation $B_x(r) := (x - r, x + r)$.

7.2.1 Characterization of Continuous Functions Using Pre-images

Theorem 1: Let $D \subseteq R$ and $f : D \to R$ a function. Then the following propositions are equivalent:

- f is continuous (on D).
- $\forall G \subseteq R$ open, $f^{-1}(G)$ is open in D.
- $\forall F \subseteq R$ closed, $f^{-1}(F)$ is closed in D.

Proof: $a \Rightarrow b$. Let $G \subseteq R$ open. Pick $a \in f^{-1}(G)$. Then $f(a) \in G$ and since G is open, there must exist $\varepsilon > 0$ such that $B_{f(a)}(\varepsilon) \subseteq G$. By continuity, corresponding to this $\varepsilon > 0$ there exists $\delta > 0$ such that $f\left(B_a(\delta) \cap D\right) \subseteq B_{f(a)}(\varepsilon)$. But this places the entire set $B_0(\delta) \cap D$ inside $f^{-1}(G)$:

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$$B_a(\varepsilon) \cap D \subseteq f^{-1}(G)$$

Writing now $\varepsilon = \varepsilon_a$ to mark the dependence of ε on a and varying $a \in f^{-1}(G)$, we obtain

$$f^{-1}(G) = \left(\bigcup_{a \in f^{-1}(G)} B_a(\varepsilon_a)\right) \cap D$$

which shows that $f^{-1}(G)$ is open in D.

$b \Leftarrow c$. Let $F \subseteq R$ a closed set, which is equivalent to saying that $G = GF$ (the complement in R) is open.

Then

$$f^{-1}(F) = \{x \in D \mid f(x) \in F\} = \{x \in D \mid f(x) \notin G\} = D - f^{-1}(G)$$

Since the complement of a D-open subset of D is D-closed, it means that $f^{-1}(F)$ is closed in D if and only if $f^{-1}(G)$ is open in D.

$c \Rightarrow a$: Left as an exercise.

Using this characterization, we can prove for example that the composition of continuous functions is a continuous function.

Proposition: Assume $f : D \to R$ is continuous, $g : E \to R$ is continuous and $f(D) \subseteq E$. Then the function $h := gof : D \to M$ defined by $h(x) = g(f(x))$ is continuous.

Proof: Let $G \subseteq R$ an open set. Then $h^{-1}(G) = f^{-1}(g^{-1}(G))$. But $g^{-1}(G) = V \cap E$, for some open set $V \subseteq R$. But then $h^{-1}(G) = f^{-1}(V \cap E) - f^{-1}(V)$ is open in D. So h is continuous.

Theorem 2: Assume $f : D \to R$ is a continuous function, such that $f(x) \neq 0$, $\forall x \in D$. Then $h : D \to R$ given by $h(x) = 1/f(x)$, is continuous as well.

Proof: $g : R - \{0\} \to R$ $g(x) = 1/x$ is continuous, $f(D) \subseteq R - \{0\}$, hence $h = gof$ is continuous.

7.2.2 General Properties of Continuous Functions

Theorem 3: A continuous function maps compact sets into compact sets.

Proof: In other words, assume $f : D \to R$ is continuous and D is compact. Then we need to prove that the image $f(D)$ is a compact subset of R. For that, we consider an arbitrary open covering $f(D) \subseteq \cup_\alpha G_\alpha$ of $f(D)$ and we will try to find a finite subcovering. Taking the preimage we have $D \subseteq \cup_\alpha f^{-1}(G_\alpha)$. But $f^{-1}(G_\alpha)$ is open in D, so there must exist $V_a \subseteq R$ open such that $f^{-1}(G_\alpha) = V_\alpha \cap D$. Then $D \subseteq \cup_\alpha (V_\alpha \cap D)$ which simply means that $D \subseteq \cup_\alpha V_\alpha$. We thus arrived at an open covering of D. So there must exist finitely many indices $\alpha_1, \ldots, \alpha_N$ such that $D \subseteq$

$\cup_{i=1}^{N} G_{\alpha_i}$, which implies the equality $D = \cup_{i=1}^{N}(V_{\alpha_i} \cap D) = \cup_{i=1}^{N} f^{-1}\left(G_{\alpha_i}\right)$. But this implies in turn that $f(D) \subset \cup_{i=1}^{N} G_{\alpha_i}$ is compact.

Theorem 4: A continuous function maps connected sets into connected sets.

In other words, assume $f: D \to R$ is continuous and D is connected. Then $f(D)$ is connected as well.

Proof: Assume $f(D)$ is not connected. Then there must exist A, B disjoint, non-empty subsets of $f(D)$, both open relative to $f(D)$, such that $f(D) = A \cup B$. Being open relative to $f(D)$ simply means there exists $U, V \subseteq R$ open such that $A = f(D) \cap U$, $B = f(D) \cap V$. So $f(D) \subseteq U \cup V$. But this implies that $D \subset f^{-1}(U) \cap f^{-1}(V)$. Since U, V are open, it follows that $f^{-1}(U)$ and $f^{-1}(V)$ are open relative to D. But they are also disjoint. Since D is connected, it follows that at least one of them, say $f^{-1}(U)$, is empty. But $A \subseteq U$, so this forces $f^{-1}(A) = \emptyset$ as well, which is impossible unless $A = \emptyset$ (note that A is a subset of the image of f), contradiction.

Theorem 5: A continuous function on a compact set is uniformly continuous.

Proof: Assume D compact and $f: D \to R$ continuous. Given $\varepsilon > 0$ we need to find $\delta(\varepsilon) > 0$ such that if $x, y \in D$ and $|x - y| < \delta(\varepsilon)$, then $|f(x) - f(y)| < \varepsilon$.

From the definition of continuity, given $\varepsilon > 0$ and $x \in D$, there exists $\delta_x(\varepsilon)$ such that if $|y - x| < \delta_x(\varepsilon)$, then $|f(y) - f(x)| < \varepsilon$. Clearly $D \subseteq \bigcup_{x \in D} B_x\left(\frac{1}{2}\delta(\varepsilon/2)\right)$. From this open covering we can extract a finite subcovering (D is compact), meaning there must exists finitely many $x_1, x_2, ..., x_N$ $\in D$ such that $D \subset \cup_{i=1}^{N} B_{x_i}\left(\frac{1}{2}\delta_{x_i}(\varepsilon/2)\right)$.

Let now $\delta(\varepsilon) = \min\{\frac{1}{2}\delta_{x_i}(\varepsilon/2)\}$.

Take $y, z \in D$ arbitrary such that $|y - z| < \delta(\varepsilon)$. The idea is that y will be near some x_j, which in turn places z near that same x_j. But that forces both $f(y)$, $f(z)$ to be close to $f(x_j)$ (by continuity at x_j), and hence close to each other.

Since $y \in D$, there must exist some j, $1 \le j \le N$ such that $y \in B_{x_j}\left(\frac{1}{2}\delta_{x_j}(\varepsilon/2)\right)$. Thus

$$\Rightarrow \quad |y - x_j| < \frac{1}{2}\delta_{x_j}(\varepsilon/2)$$

õ but $|y - z| <$ ¢(£) á $\frac{1}{2}$ ¢$_{x_j}$ (£/2)

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By the triangle inequality it follows that $|z - x| <$ ¢$_{x_j}$ (£/2). So y, z are within ¢$_{x_j}$ (£/2) of x.

This implies that

õ $|f(y) - f(x_j)| <$ £/2

õ $|f(z) - f(x_j)| <$ £/2

By the triangle inequality once again we have $|f(y) - f(z)| <$ £.

7.2.3 Alternative Proof Using Sequences

Assume f is not uniformly continuous, meaning that there exists £ > 0 such that no ¢ > 0 does the job. Checking what this means for ¢ = $\frac{1}{n}$, we see that for any such 3n ñ 1 there exist x_n, y_n Č D such that $|x_n - y_n| < \frac{1}{n}$ and yet $|f(x_n) - f(y_n)| >$ £. However D is compact; in particular any sequence in D has a convergent subsequence whose limit belongs to D. Applying this principle twice we find that there must exist $n_1 < n_2 < ...$ such that the subsequences $\left(x_{n_k}\right)_{k\text{ñ}1}$ and $\left(y_{n_k}\right)_{k\text{ñ}1}$ are convergent, and $x = \lim_{x\text{ ì ã}} x_{n_k}$ Č D, $y = \lim_{x\text{ ì ã}} y_{n_k}$ Č D. We have the following:

õ By construction, $\left|x_{n_k} - y_{n_k}\right|$ Y $\frac{1}{n_k}$ á $\frac{1}{k}$. Taking the limit, we find $x = y$.

õ By continuity, $\lim_{k\text{ ì ã}} f\left(x_{n_k}\right)$ Z $f(x)$, since x Č D. Also

$\lim_{k\text{ ì ã}} f\left(y_{n_k}\right)$ Z $f(y)$.

õ Also by construction, $\left|f\left(x_{n_k}\right) - \left(y_{n_k}\right)\right| >$ £. Hence in the limit, $|f(x) - f(y)|$ > £.

We thus reach a contradiction.

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Proposition: Let $D \subseteq R$. Then the following propositions are equivalent:

a) D is compact

b) D is bounded and closed

c) Every sequence in D has a convergent subsequence whose limit belongs to D.

Proof: $a \Rightarrow b$. $D \subseteq R = \bigcup_{n=1}^{\infty}(-n, n)$ is an open covering of that D. Hence $\exists N \geq 1$ such that $D \subseteq \bigcup_{n=1}^{\infty}(-n, n) = (-N, N)$. This shows that D is bounded. To prove D is closed, we prove that $R - D$ is open. Let $y \in R - D$. Then $D \subseteq \bigcup_{n=1}^{\infty}\left\{R - [y - \frac{1}{n}, \frac{1}{n}]\right\}$. This open covering must have a finite subcovering, so $\exists N \geq 1$ such that $D \subseteq R - [y - \frac{1}{N}, y + \frac{1}{N}]$. But this implies that $\left\{y - \frac{1}{N}, y + \frac{1}{N}\right\} \subseteq R - D$. But y was chosen arbitrary in $R - D$, so this set is open, and hence D itself is closed.

$b \Rightarrow c$. This has to do with the fact that every bounded sequence has a convergent subsequence.

$c \Rightarrow b$. Here one shows $D = \bar{D}$ and this has to do with the fact that D is the set of limits of convergent sequences of D, etc.

$c \Rightarrow a$. Let $D \subseteq \cup_{k=1}^{\infty} G_k$ be an arbitrary open covering of D.

Note: A covering by a countable collection of open sets is not the most general infinite open covering one can imagine, of course; we need an intermediate step to prove that from any open covering of D we can extract a countable subcovering, and this has to do with the fact that R admits a countable dense set.

We will now prove that there exists $n \geq 1$ such that $D \subseteq \cup_{k=1}^{\infty} G_k$. Assume this was not the case. Then $\forall N \geq 1$, there exists $x_n \in D - \cup_{k=1}^{\infty} G_k$. But x_n is a sequence in D, so it must have a convergent subsequence; call it $\left(x_{n_j}\right)_{j \geq 1}$, with limit in D. So $\lim_{j \to \infty} x_{n_j} = a \in D$. But a belongs to one of the G_i's, say $a \in G_N$. Since G_N is open, it follows that $x_{n_j} \in G_N$, for $j \geq j_0$ (j large enough). In particular this shows that for j large enough (larger than j_0 and larger than N) we have $x_{n_j} \in G_N \subset \bigcup_{k=1}^{n_j}$, since $n_j \geq j > N$. This contradicts the defining property of x_n's.

Theorem 6: R is connected.

Proof: This can be restated as, $\emptyset$ and R itself are the only subsets of R which are both open and closed. To prove this, let E be a non-empty subset of R with this property. We will prove that $E = R$. For that, take an arbitrary $c \in R$. To prove that $c \in E$, we assume that $c \notin E$ and look for a contradiction. Since E is non-empty, it follows that E either has points to the left of c or to the right of c. Assume that the former holds.

- Consider the set $S = \{x \in E | x < c\}$. By construction, S is bounded from above (c is an upper bound for S). Therefore we can consider $y = lub(S) \in R$.
- Input: E is closed. Then $S = E \cap (-\infty, c]$ is also closed. Then $y \in \overline{S} = S$, so $y < c$.
- Input: E is open, $y \in S \subseteq E$ and E is open. This means that there exists $\varepsilon > 0$ such that $(y - \varepsilon, y + \varepsilon) \subseteq E$. Choose ε small enough so that $\varepsilon < c - y$. In that case $z = y + \varepsilon/2 \in (y - \varepsilon, y + \varepsilon) \subseteq E$ is an element of E with the properties,
  - $z < c$, hence $z \in S$
  - $z > y$

which is in contradiction with the defining property of y.

Theorem 7: The only connected subsets of R are the intervals (bounded or unbounded, open or closed or neither).

Proof: First we prove that a connected subset of R must be an interval.

Let $E \subseteq R$ be a connected subset. We prove that if $a < b \in E$, then $[a,b] \subseteq E$. In other words, together with any two elements, E contains the entire interval between them. To see this, let c be a real number between a and b. Assume $c \notin E$. Then $E = A \cup B$, where $A = (-\infty, c) \cap E$ and $B = (c, +\infty) \cap E$. Note that A and B are disjoint subsets of D, both open relative to D. Since S is connected, at least one of them should be empty, contradiction, since $a \in A$ and $b \in B$. Thus $c \in E$.

To show that E is actually an interval, consider inf E and sup E. E is bounded. Then $m = \inf E$, $M = \sup E \in R$, and clearly $E \subseteq [m, M]$. On the other hand, for any given $x \in (m, M)$, there exists $a,b \in E$ such that $a < x < b$. That is because m, $M \in E$ and one can find elements of E as close to m (respectively M) as desired (draw a picture with the interval (m, M) and place a point x inside it). But then $[a,b] \subseteq E$, and in particular $x \in E$. Since x was chosen arbitrarily in (m, M), we must have $(m, M) \subseteq E \subseteq [m, M]$, so E is definitely an interval. Case two: E is unbounded. With a similar argument, show that E is an unbounded interval.

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Conversely, we need to show that intervals are indeed connected sets. The proof is almost identical to that in the case where the interval is R itself.

Theorem 8: Let $D \subseteq R$ be compact and $f : D \to R$ be a continuous function. Then there exists $y_1, y_2 \in D$ such that $f(y_1) \leq f(x) \leq f(y_2)$, $\forall x \in D$.

Proof: $f(D)$ is a compact subset of R, so it is bounded and closed. This implies that glb$(f(D)) \in f(D)$ and lub$(f(D)) \in f(D)$ as well. But then there must exist $y_1, y_2 \in D$ such that $f(y_1) =$ glb $f(D)$ and $f(y_2) =$ lub $f(D)$. But this implies $f(D) \subseteq [f(y_1), f(y_2)]$ and we are done.

Note: We use the notation $\sup_{x \in D} f(x)$ to denote the lub of the image of D. In other words, $\sup_{x \in D} f(x) = \text{lub}\{f(y) \mid y \in D\}$. The theorem says that if D is compact and f is continuous, then $\sup_{x \in D} f(x)$ is finite, and moreover that there exists $y_1 \in D$ such that $f(y_1) = \sup_{x \in D} f(y)$. If the domain is not compact, one can find examples of continuous functions such that either i) sup $f = +\infty$ or such that ii) sup f is a real number but not in the image of f.

For case i), take $f(x) = 1/x$ defined on $(0,1]$. For case ii), take $f(x) = x$ defined on $[0,1)$.

Theorem 9: A continuous (real-valued) function defined on an interval in R has the intermediate value property.

Proof: Assume E is an interval in R and $f: E \to R$ a continuous function. Let $a,b \in E$ (say $a < b$) and y a number between $f(a)$ and $f(b)$. The intermediate value property is the statement that there exists c between a and b such that $f(c) = y$. But this follows immediately from the fact that $f(E)$ is an interval. E is an interval in R $\Rightarrow$ E is connected $\Rightarrow$ $f(E)$ is a connected subset of $R \Rightarrow f(E)$ is an interval in R.

7.3 HOMEOMORPHISM

Topological equivalence redirects here. There is no need of a continuous deformation for two spaces to be homeomorphic.In the mathematical field of topology, a homeomorphism or topological isomorphism (from the Greek words (homoios)' means similar and (morphe)' means shape which is taken form Latin deformation of morphe is a bicontinuous function between two topological spaces. Homeomorphisms are the isomorphisms in the category of topological spaces, that is, they are the mappings which preserve all the topological properties of a given space. Two spaces with a homeomorphism between them are called homeomorphic, and from a topological viewpoint they are the same.

A topological space is a geometric object, and the homeomorphism is a continuous stretching and bending of the object into a new shape. Thus, a square and a circle are homeomorphic to each other, but a sphere and a donut are not.

An often-repeated joke is that topologists can not tell the coffee cup from which they are drinking from the donut they are eating, since a sufficiently pliable donut could be reshaped to the form of a coffee cup by creating a dimple and progressively enlarging it, while shrinking the hole into a handle.

A homeomorphism maps points in the first object that are 'close together' to points in the second object that are close together and points in the first object that are not close together to points in the second object that are not close together. Topology is the study of those properties of objects that do not change when homeomorphisms are applied.

A function '*f*' between two topological spaces '*X*' and '*Y*' is called a homeomorphism if it has the following properties:

- '*f*' is a bijection (1-1 and onto).
- '*f*' is continuous.
- The inverse function 'f' $^{-1}$ is continuous (*f* is an open mapping).

A function with these three properties is sometimes called bicontinuous. If such a function exists, we say '*X*' and '*Y*' are homeomorphic. A self-homeomorphism is a homeomorphism of a topological space and itself. The homeomorphisms form an equivalence relation on the class of all topological spaces. The resulting equivalence classes are called homeomorphism classes. For example,

- The unit 2-disc $\mathbf{D}^2$ and the unit square in $\mathbf{R}^2$ are homeomorphic.
- The open interval (–1, 1) is homeomorphic to the real numbers **R**.
- The product space S^1 and the two-dimensional torus are homeomorphic.
- Every uniform isomorphism and isometric isomorphism is a homeomorphism.
- Any 2-sphere with a single point removed is homeomorphic to the set of all points in $\mathbf{R}^2$ (a 2-dimensional plane).
- Let *A* be a commutative ring with unity and let *S* be a mutiplicative subset of *A*.
- An example of a continuous bijection that is not a homeomorphism is the map that takes the half-open interval [0,1) and wraps it around the circle. In this case the inverse — although it exists — fails to be continuous. The primage of certain sets which are actual open in the relative topology of the half-open inteval are not open in the more natural topology of the circle (they are half-open intevals).

Homeomorphisms are the isomorphisms in the category of topological spaces. As such, the composition of two homeomorphisms is again a homeomorphism and the set of all self-homeomorphisms '*X*' forms a group called the homeomorphism group of '*X*'.

For some purposes, the homeomorphism group happens to be too big, but by means of the isotopy relation, one can reduce this group to the mapping class group.

NOTES

1. Two homeomorphic spaces share the same topological properties. For example, if one of them is compact, then the other is as well; if one of them is connected, then the other is as well; if one of them is Hausdorff, then the other is as well; their homology groups will coincide. Note however that this does not extend to properties defined via a metric; there are metric spaces which are homeomorphic even though one of them is complete and the other is not.
2. A homeomorphism is simultaneously an open mapping and a closed mapping, that is it maps open sets to open sets and closed sets to closed sets.
3. Every self-homeomorphism in S^1 can be extended to a self-homeomorphism.

Check Your Progress

1. Write any two general properties of a continuous functions.
2. What does homeomorphism mean in topology?
3. A square and a circle are homeomorphic to each other, but a sphere and a donut are not. Why?
4. Give an example of a continuous bijection that is not a homeomorphism.

7.4 THE PRODUCT TOPOLOGY

In topology and related fields of mathematics, there are many limitations that we often make on the kinds of topological spaces that we wish to consider. Some of these restrictions are given by the *separation axioms*. These separation axioms are often termed as *Tychonoff separation axioms*, after Andrey Tychonoff. The separation axioms are axioms only in the sense that, when defining the notion of topological space, one could add these conditions as extra axioms to get a more restricted notion of what a topological space is. The separation axioms are denoted with the letter '*T*' after the German *Trennungsaxiom*, meaning separation axiom.

The separation axioms are used to distinguish disjoint sets and distinct points. It is not enough for elements of a topological space to be distinct. They have to be *topologically distinguishable*. In the same way, it is not enough for subsets of a topological space to be disjoint. They have to be separated in some way. Suppose X is a topological space. Then the two points x and y in X are topologically distinguishable if they do not have exactly the same neighbourhoods or we can say that at least one of them has a neighbourhood that is not a neighbourhood of

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the other. If x and y are topologically distinguishable points, then the singleton sets $\{x\}$ and $\{y\}$ must be disjoint.

Two points x and y are *separated* if each of them has a neighbourhood that is not a neighbourhood of the other. More generally, two subsets A and B of X are *separated* if each is disjoint from the other's closure; the closures themselves do not have to be disjoint. The points x and y are separated if and only if their singleton sets $\{x\}$ and $\{y\}$ are separated. All of the remaining conditions for sets may also be applied to points or to a point and a set by making use of singleton sets.

Additionally, subsets A and B of X are separated by neighbourhoods if they have disjoint neighbourhoods. They are separated by closed neighbourhoods if they have disjoint closed neighbourhoods. They are separated by a function if there exists a continuous function f from the space X to the real line $\mathbf{R}$ such that the image $f(A)$ equals $\{0\}$ and $f(B)$ equals $\{1\}$. Lastly, they are precisely separated by a function if there exists a continuous function f from X to $\mathbf{R}$ such that the preimage $f^{-1}(\{0\})$ equals A and $f^{-1}(\{1\})$ equals B. Any two topologically distinguishable points must be distinct and any two separated points must be topologically distinguishable. Moreover, any two separated sets must be disjoint and any two sets separated by neighbourhoods must be separated, and so on.

Theorem 10: A topological space X is Hausdorff if and only if the diagonal is closed in $X \times X$ with the product topology.

Proof: Let D denote the diagonal $\{(x, x) \mid x \in X\}$ in $X \times X$.

Suppose D is closed. Then the complement of D is open. We want to show that X is Hausdorff. Let $x \neq y$. The point (x, y) lies in an open set disjoint from D. In particular, there is a basis open set about (x, y) that does not intersect D. Let $U \times V$ be such a basis open set, so U and V are both open in X. Clearly, if $y \in U \cap V$, then $(y, y) \in U \times V$. But $U \times V$ does not intersect D, and hence U and V are disjoint open sets.

Conversely, suppose X is Hausdorff. Then, we want to show that D is closed.

We know that given $x \neq y$, there are disjoint open sets U and V containing x and y respectively. Thus, any (x, y) lies inside an open set $U \times V$. Further, as $U \cap V$ is empty, the set $U \times V$ does not intersect D. Hence, for every point outside D, there is a neighbourhood of the point outside D. Taking the union of all these neighbourhoods, we conclude that the complement of D is open, and hence that D is closed.

Though the two directions of proof seem almost identical to one another, there is a subtle difference. In proving that the diagonal is closed from Hausdorffness, all we need to use is that products of open sets are open. Thus, the same proof will go through if we add more open sets. On the other hand, the proof of Hausdorffness from the diagonal being closed critically uses the fact that the so called open rectangles Ethe products of open setsF-form a basis.

Theorem 11: The product of Hausdorff spaces is Hausdorff in the product topology.

Proof: Let X and Y be the two Hausdorff spaces. Then, the product space is X ò Y.

Select the points (x_1, y_1) and (x_2, y_2). Now either $x_1 \div x_2$ or $y_1 \div y_2$.

If $x_1 \div x_2$, first separate x_1 and x_2 in X. That is, choose disjoint open sets in $X : U_1$ containing x_1 and U_2 containing x_2. Clearly $U_1 \times Y$ and $U_2 \times Y$ are disjoint open sets containing (x_1, y_1) and (x_2, y_2) respectively.

If $y_1 \div y_2$, then separate y_1 and y_2 in Y, say, by V_1 and V_2. Then $X \times V_1$ and $X \times V_2$ are disjoint open sets containing (x_1, y_1) and (x_2, y_2).

Theorem 12: Every subspace of a regular space is regular.

Proof: Let X be the regular topological space and A be a subset. Choose x Č A and C closed in A. For x Č X, choose D, a closed subset of X such that D ą A Z C. Such a D exists from the way the subspace topology is defined. Clearly, whatever D is picked up for the purpose, x č D because the only points in $D \cap A$ are in a set not containing x.

Since X is regular, we can find open sets U and V in X such that x Č U, $C \subseteq V$, and U and V are disjoint. Now, $U \cap A$ and $V \cap A$ are disjoint open subsets of A, with x Č U ą A and C ċ V ą A.

The two important facts that we chose are that x being a point in the subspace, remained a point in the whole space and every closed set in the subspace was an intersection with the subspace of a closed set in the whole space.

Theorem 13: A product space $\prod_{\lambda} X_{\lambda}$ is first axiom iff each space X_a is first axiom and all but a countable number are indiscrete.

Proof: Let $\prod_{\lambda} X_{\lambda}$ be first axiom. Since each space X is homeomorphic to set E where each

$$E = \left\{ \begin{array}{l} (x_\lambda) \text{ such that } x_\lambda = z_\lambda \text{ if } \lambda \neq \beta \\ x_\beta \text{ is any point of } X_\beta \text{ and} \\ (z_\lambda) \text{ is a fixed point of } X = \prod_\lambda X_\lambda \end{array} \right.$$

and the first axiom of countability is both hereditary and topological, each space is first axiom. Now if the space X_α is indiscrete, we may choose a point $x_\alpha \in X_\alpha$ which is contained in an open set $G_\alpha \neq X_\alpha$. If X_α is indiscrete, we choose any point $x_\alpha \in X_\alpha$. Let $X = (x_\alpha)$ and suppose $\{B_n\}_{n \in \mathbf{N}}$ is a countable open base at X. For each integer n, the set B_n must contain a member of the Tichonov base of the form $\prod_\lambda Y_\lambda$ where Y_α is open in X_α for all α and $Y_\alpha = X_\alpha$ for all but a finite number of values of α, $\alpha^n{}_1, \alpha^n{}_2, \ldots, \lambda^n{}_{k_n}$. The collection of all the expected values of α, $\{i = 1, 2, \ldots, k_n$ and $n \in \mathbf{N}\}$ is countable. For any other value of α, we choose $x_\alpha \in G_\alpha \neq X_\alpha$ if X_α was not discrete, but $\pi_\lambda^{-1}(G_\lambda)$ would then be an open set containing X which would not contain any B_n. Hence for all other values of α, X_α must be indiscrete.

Now suppose that X_α is first axiom for all α and indiscrete for $\alpha \notin \{\alpha_i\}_{i \in \mathbf{N}}$. Let $X = (x_\alpha)$ be an arbitrary point in $\prod_\lambda X_\lambda$, and let $\{B_{n,\alpha}\}$ be a countable open base at x_α. We note that $B_{n,\alpha} = x_\alpha$ for all n if $\alpha \notin \{\alpha_i\}$. The family $\left\{\pi^{-1}{}_{\lambda_i}(B_n, \lambda_i), i, n \in \mathbf{N}\right\}$ is a countable collection of open sets in the product space. The set of all finite intersections of members of this collection is also countable and it is already an open base at x, as required.

Check Your Progress
5. When are two points topologically distinguishable?
6. When is a topological space Hausdorff?

7.5 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1. (i) A continuous function maps compact sets into compact sets.

   (ii) A continuous function maps connected sets into connected sets.

2. In the mathematical field of topology, a homeomorphism or topological isomorphism (from the Greek words (homoios)' means similar and (morphe)' means shape which is taken form Latin deformation of morphe is a bicontinuous function between two topological spaces.
3. A topological space is a geometric object, and the homeomorphism is a continuous stretching and bending of the object into a new shape. Thus, a square and a circle are homeomorphic to each other, but a sphere and a donut are not.
4. An example of a continuous bijection that is not a homeomorphism is the map that takes the half-open interval [0,1) and wraps it around the circle.
5. the two points x and y in X are topologically distinguishable if they do not have exactly the same neighbourhoods or we can say that at least one of them has a neighbourhood that is not a neighbourhood of the other. If x and y are topologically distinguishable points, then the singleton sets $\{x\}$ and $\{y\}$ must be disjoint.
6. A topological space X is Hausdorff if and only if the diagonal is closed in $X \times X$ with the product topology.

7.6 SUMMARY

- $D \subseteq R$ is compact if and only if for any given open covering of D we can subtract a finite sucovering. That is, given $(G_\alpha)_{\alpha \in A}$, a collection of open subsets of R (A an arbitrary set of indices) such that $D \subset \cup_{\alpha \in A} G_\alpha$, then there exist finitely many indices $\alpha_1, ..., \alpha_n \in A$ such that $D \subset \cup_{i=1}^{N} G_{\alpha_1}$.
- $D \subset R$ is connected if and only if $\emptyset$ and D are the only subsets of D which are both open in D and closed in D. In other words, if $D = A \cup B$ and A, B are disjoint D-open subsets of D, then either $A = \emptyset$ or $B = \emptyset$
- Assume $f : D \to R$ is a continuous function, such that $f(x) \neq 0$, $\forall x \in D$. Then $h : D \to R$ given by $h(x) = 1/f(x)$, is continuous as well.
- A continuous function on a compact set is uniformly continuous.
- Let $D \subseteq R$. Then the following propositions are equivalent:

  a) D is compact

  b) D is bounded and closed

  c) Every sequence in D has a convergent subsequence whose limit belongs to D.
- The only connected subsets of R are the intervals (bounded or unbounded, open or closed or neither).

- Let $D \subseteq R$ be compact and $f : D \to R$ be a continuous function. Then there exists $y_1, y_2 \in D$ such that $f(y_1) \leq f(x) \leq f(y_2)$, $\forall x \in D$.
- A continuous (real-valued) function defined on an interval in R has the intermediate value property.
- Homeomorphisms are the isomorphisms in the category of topological spaces, that is, they are the mappings which preserve all the topological properties of a given space.
- A homeomorphism maps points in the first object that are 'close together' to points in the second object that are close together and points in the first object that are not close together to points in the second object that are not close together.
- *A* function '*f*' between two topological spaces '*X*' and '*Y*' is called a homeomorphism if it has the following properties:
  - '*f*' is a bijection (1-1 and onto).
  - '*f*' is continuous.
  - The inverse function 'f^{-1}' is continuous (*f* is an open mapping).

  A function with these three properties is sometimes called bicontinuous.
- Separation axioms are often termed as *Tychonoff separation axioms*, after Andrey Tychonoff.
- The separation axioms are denoted with the letter '*T*' after the German *Trennungsaxiom*, meaning separation axiom.
- The separation axioms are used to distinguish disjoint sets and distinct points.
- two subsets *A* and *B* of *X* are *separated* if each is disjoint from the other's closure; the closures themselves do not have to be disjoint.
- Every subspace of a regular space is regular.
- A product space $\prod_{\lambda} X_{\lambda}$ is first axiom iff each space X_{λ} is first axiom and all but a countable number are indiscrete.

7.7 KEY WORDS

- **Continuous function:** A function for which sufficiently small changes in the input result in arbitrarily small changes in the output is known as a continuous function.
- **Homeomorphism:** A continuous function with a continuous inverse function is known as a homeomorphism.

- **Product topology:** A product space is the Cartesian product of a family of topological spaces equipped with a natural topology known as the product topology.
- **Separation axioms:** Separation axioms are used to to distinguish disjoint sets and distinct points. They are denoted with the letter “T”

7.8 SELF ASSESSMENT QUESTIONS AND EXERCISES

Short Answer Questions

1. Write a short note on characterization of continuous functions using pre-images.
2. Describe general properties of continuous functions briefly.
3. Give an alternative proof using sequences to show that a continuous function on a compact set is uniformly continuous.
4. Show that a topological space X is Hausdorff if and only if the diagonal is closed in $X \times X$ with the product topology.

Long Answer Questions

1. Discuss continuous functions and continuity of a function giving examples.
2. Discuss homeomorphisms with suitable examples.
3. Give a detailed account of the product topology.

7.9 FURTHER READINGS

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UNIT 8 METRIC AND QUOTIENT TOPOLOGY

NOTES

Structure

8.0 INTRODUCTION

A metric space is a kind of topological space. In a metric space any union of open sets is open and any finite intersection of open sets is open. Consequently a metric space meets the axiomatic requirements of a topological space and is thus a topological space. This particular property of a metric space was used to define a topological space. A quotient space, also known as an identification space, is the result of identifying certain points of a given topological space. The quotient topology consists of all sets with an open pre-image under the canonical projection map that maps each element to its equivalence class. In this unit you will study construction of continuous functions. You will describe the metric topology and their properties. The quotient topology is also explained in this unit.

8.1 OBJECTIVES

After going through this unit, you will be able to:

- Understand the construction of continuous functions
- Describe the metric topology and their properties
- Explain quotient topology

8.2 CONSTRUCTING CONTINUOUS FUNCTIONS

Continuity of functions is one of the core concepts of topology. The special case is defined as where the inputs and outputs of functions are real numbers. A stronger form of continuity is uniform continuity. Any function from a discrete space to any

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other topological space is continuous and also any function from any topological space to an indiscrete space is continuous. Any constant function is continuous (regardless of the topologies on the two spaces). The pre-image under such a function of any set containing the constant value is the whole domain, and the pre-image of any set not containing the constant value is empty.

Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^3$. Then f is a continuous function from $\mathbb{R}_{\text{usual}}$ to $\mathbb{R}_{\text{usual}}$.

Rules for Constructing Continuous Functions

Let X, Y, and Z be topological spaces.

1. **Constant Function:** If $f : X \to Y$ maps all of X into a single point $y_0 \in Y$, then f is continuous.
2. **Inclusion:** If A is a subspace of X, the inclusion function $j : A \to X$ is continuous.
3. **Composites:** If $f : X \to Y$ and $g : Y \to Z$ are continuous, then the map $g \circ f : X \to Z$ is continuous.
4. **Restricting the Domain:** If $f : X \to Y$ is continuous and if A is a subspace of X, then the restricted function $f|_A : A \to Y$ is continuous.
5. **Restricting or Expanding the Range:** Let $f : X \to Y$ be continuous. If X is a subspace of Y containing the image set $f(X)$, then the function $g : X \to Z$ obtained by restricting the range of f is continuous. If Z is a space having Y as a subspace, then the functions $h : X \to Z$ obtained by expanding the range of f is continuous.
6. **Local Formulation of Continuity:** The map $f : X \to Y$ is continuous if X can be written as the union of open sets U_α such that $f|_{U\alpha}$ is continuous for each α.

Check Your Progress

1. Give definition of a constant *t* function.
2. What do you understand by restricting the domain?

8.3 THE METRIC TOPOLOGY

Let X be a non-empty set. A function d from $X \times X$ into **R** (the set of reals) is called a metric (or distance function) if for all $x, y, z \in X$, the following conditions are satisfied.

[**m 1**]: $d(x, y) \geq 0$.

[**m 2**]: $d(x, y) = 0$ if and only if $x = y$.

[**m 3**]: $d(x, y) = d(y, x)$. **(Symmetry)**

[**m 4**]: $d(x, z) \leq d(x, y) + d(y, z)$. **(Triangle Inequality)**

(i) The pair (X, d) is called a metric space and $d(x, y)$ is called the distance between the points x and y.

Another definition is, the diameter of a subset A of a metric space X, denoted by $\delta(A)$, is defined by,

$\delta(A) = \sup\{d(x, y): x, y \in A\}$.

(ii) The distance between two subsets A, B, of X, denoted by $d(A, B)$, is defined by,

$d(A, B) = \inf\{d(x, y): x \in A, y \in B\}$.

(iii) The distance between a point $a \in X$ and a set $A \subset X$ is defined by,

$d(a, A) = \inf\{d(a, x): x \in A\}$.

(iv) A subset A of X is said to be bounded if $\delta(A)$ is finite. It follows that A is bounded if there exists a real number M and a point $q \in X$ such that $d(p, q) < M$ for all $p \in A$.

The most important examples of metric spaces, for our purposes, are the Euclidean spaces $\mathbf{R}^n$, in particular the real line $\mathbf{R}$ and the complex plane $\mathbf{R}^2$. The distance in $\mathbf{R}^n$ is defined by,

$$d(x, y) = |x - y| \quad (x, y \in \mathbf{R}^n). \qquad ...(1.1)$$

The conditions [m1], [m2], [m3] and [m4] are satisfied by Theorem 1.1 and thus $\mathbf{R}^n$ is a metric space.

Another example is Let X be any non-empty set. For $x, y \in X$, define

$$d(x, y) = \begin{cases} 0 \text{ if } x = y \\ 1 \text{ if } x \neq y. \end{cases}$$

Then it is easy to see that d is a metric on X called the discrete metric.

It can also be defined as,

(i) Let $a_i < b_i$ for $i = 1, ..., n$. Then the set of all points $x = (x_i, ..., x_n)$ in $\mathbf{R}^n$ whose coordinates satisfy the inequalities $a_i \le x_i \le b_i$ $(1 \le i \le n)$ is a called n-cell.

Thus 1-cell is an interval and 2-cell is a rectangle, etc.

(ii) Let a $\in \mathbf{R}^n$ and let $r > 0$. An open (or closed) ball with centre at a and radius r is defined to be set of all $x \in \mathbf{R}^n$ such that $|x - a| < r$ (*or* $|x - a| \le r$ and shall be denoted by,

$B(a, r)$ (or $B[a, r]$), Thus

$B(a, r) = \{x \in \mathbf{R}^n: |x - a| < r\}$

and $B[a, r] = \{x \in \mathbf{R}^n: |x - a| \le r\}$,

(iii) A subset A *of* $\mathbf{R}^n$ is called to be convex if,

$\lambda x + (1 - \lambda) y \in A$

whenever $x \in A$, $y \in A$ and $0 < \lambda < 1$.

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A metric space is a kind of topological space. In a metric space any union of open sets is open and any finite intersection of open sets is open. Consequently a metric space meets the axiomatic requirements of a topological space and is thus a topological space. It was, in fact, this particular property of a metric space that was used to define a topological space.

Theorem 1: The collection of open spheres in a set X with metric d is a base for a topology on X.

Proof : Let d be a metric on a non-empty set X. The topology τ on X generated by the collection of open spheres in X is called the **metric topology**.

The set X together with the topology x induced by the metric d is a metric space. A metric space then can be viewed as a topological space in which the topology is induced by a metric.

Lemma 1: Let d be the usual metric in three dimensional space $\mathbf{R}$. Then the set of open spheres in $\mathbf{R}^3$ constitute a base for a topology on $\mathbf{R}^3$. Thus the usual metric on $\mathbf{R}^3$ induces the usual topology on $\mathbf{R}^3$, the collection of all open sets.

Lemma 2: Let d be the usual metric on the real line $\mathbf{R}$, i.e., $d(a, b) = |a - b|$. Then the open spheres in $\mathbf{R}$ correspond to the finite open intervals in $\mathbf{R}$. Thus the usual metric on $\mathbf{R}$ induces the usual topology, the set of all open intervals, on $\mathbf{R}$.

Lemma 3: Let d be the trivial metric on some set X. Note that for any $p \,\varepsilon\, X$, $S(p, 1/2) = \{p\}$. Thus every singleton set (set consisting of only one element) is open and consequently every set is open. Hence the trivial metric induces the discrete topology on X.

Properties of Metric Topologies

Theorem 2: Let p be a point is a metric space X. Then the countable class of open spheres $\{S(p, 1), S(p, 1/2), S(p, 1/3), S(p, 1/4), ...\}$ is a local base at p.

Theorem 3: The closure $\overline{A}$ of a subset A of a metric space X is the set of points whose distance from A is zero.

In a metric space all singleton sets $\{p\}$ are closed.

Theorem 4: In a metric space, all finite sets are closed.

Following is an important 'separation' property of metric spaces.

Theorem 5: (Separation Axiom). Let A and B be closed disjoint subsets of a metric space. Then there exist disjoint open sets G and H such that $A \subset G$ and $B \subset H$.

One might think that the distance between two disjoint closed sets would be greater than zero. However, this is not necessarily the case as the following example shows.

For example, the two sets:

$$A = \{(x, y): xy \geq 1, x < 0\}$$
$$B = \{(x, y): xy \geq 1, x > 0\}$$

are closed and they are disjoint. However, $d(A, B) = 0$.

Consider another example where let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two arbitrary points in the plane $\mathbf{R}^2$. The usual metric d and the two metrics d_1 and d_2 defined by,

$$d_1(P, Q) = \max (|x_1 - x_2|, |y_1 - y_2)$$
$$d_2(P, Q) = |x_1 - x_2| + |y_1 - y_2|$$

All Induce the usual topology on the plane $\mathbf{R}^2$ since the collection of open spheres of each metric is a base for the usual topology on $\mathbf{R}^2$. The open spheres of each of the three metrics. Thus, the three metrics are equivalent.

Isometric Metric Spaces

A metric space (X, d) is isometric to a metric space (Y, e) if and only if there exists a one-to-one onto function $f: X \rightarrow Y$ which preserves distances, i.e., for all $p, q \in X$,

$$d(p, q) = e(f(p), f(q)$$

Theorem 6: If the metric space (X, d) is isometric to (Y, e), then (X, d) is also homeomorphic to (Y, e).

8.4 THE QUOTIENT TOPOLOGY

In topology and mathematics, a **quotient space**, also called an identification space, is the result of identifying certain points of a given topological space. The points to be identified are specified by an **equivalence relation**. This is commonly done in order to construct new spaces from given ones. The quotient topology consists of all sets with an open pre-image under the canonical projection map that maps each element to its equivalence class.

Definition

Let (X, τ_X) be a topological space, and let $\sim$ be an equivalence relation on X. The quotient space, $Y = X/\sim$ is defined to be the set of equivalence classes of elements of X as follows:

$$Y = \Big\{[x] : x \in X\Big\} = \Big\{\{v \in X : v \sim x\} : x \in X\Big\}$$

This is equipped with the topology where the open sets are defined to be those sets of equivalence classes whose unions are open sets in X:

$$\tau_Y = \left\{U \subseteq Y : \bigcup U = \left(\bigcup_{[a]\in U}[a]\right) \in \tau_X\right\}$$

Equivalently, we can define them to be those sets with an open pre-image under the surjective map $q : X’‡ X/\sim$, which sends a point in X to the equivalence class containing it:

$$\tau_Y = \left\{U \subseteq Y : q^{-1}(U) \in \tau_X\right\}$$

The quotient topology is the final topology on the quotient space with respect to the map q.

Check Your Progress

3. State the separation axiom.
4. What is meant by isometric metric space?
5. Give definition for the quotient space?

8.5 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1. If $f : X \to Y$ maps all of X into a single point $y_0 \in Y$, then f is continuous.
2. If $f : X \to Y$ is continuous and if A is a subspace of X, then the restricted function $f|_A : A \to Y$ is continuous.
3. Let A and B be closed disjoint subsets of a metric space. Then there exist disjoint open sets G and H such that $A \subset G$ and $B \subset H$.
4. A metric space (X, d) is isometric to a metric space (Y, e) if and only if there exists a one-to-one onto function $f: X \to Y$ which preserves distances, i.e., for all $p, q \in X$,

   $d(p, q) \quad = e(f(p), f(q))$
5. Let (X, τ_X) be a topological space, and let $\sim$ be an equivalence relation on X. The quotient space, $Y = X/\sim$ is defined to be the set of equivalence classes of elements of X as follows:

$$Y = \left\{[x] : x \in X\right\} = \left\{\{v \in X : v \sim x\} : x \in X\right\}$$

8.6 SUMMARY

- Any function from a discrete space to any other topological space is continuous and also any function from any topological space to an indiscrete space is continuous.

- The map $f : X \to Y$ is continuous if X can be written as the union of open sets U_α such that $f|_{U\alpha}$ is continuous for each α.
- A subset A of X is said to be bounded if $\delta(A)$ is finite. It follows that A is bounded if there exists a real number M and a point $q \in X$ such that $d(p, q) < M$ for all $p \in A$.
- The collection of open spheres in a set X with metric d is a base for a topology on X.
- Let d be the usual metric in three dimensional space **R**. Then the set of open spheres in $\mathbf{R}^3$ constitute a base for a topology on $\mathbf{R}^3$. Thus the usual metric on $\mathbf{R}^3$ induces the usual topology on $\mathbf{R}^3$, the collection of all open sets.
- In topology and mathematics, a quotient space, also called an identification space, is the result of identifying certain points of a given topological space. The points to be identified are specified by an equivalence relation.

8.7 KEY WORDS

- **Metric space:** A metric space is a set together with a metric on the set.
- **Quotient space:** A quotient space (also called an identification space) is the result of identifying certain points of a given topological space.
- **Equivalence relation:** It is a binary relation that is reflexive, symmetric and transitive. The relation "is equal to" is the canonical example of an equivalence relation.

8.8 SELF ASSESSMENT QUESTIONS AND EXERCISES

Short Answer Questions

1. Describe the rules for construction of continuous functions.
2. Write properties of metric topologies.

Long Answer Questions

1. Explain the method of constructing continuous functions giving its rules and examples.
2. Give a detailed account of the metric topology.
3. Discuss the importance of quotient topology.

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8.9 FURTHER READINGS

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BLOCK - III

CONNECTED AND COMPACT SPACES

NOTES

UNIT 9 CONNECTED SPACES AND SETS

Structure

9.0 INTRODUCTION

A connected space is a topological space that cannot be represented as the union of two or more disjoint nonempty open subsets. Connectedness is one of the principal topological properties that are used to distinguish topological spaces. In general topology, compactness is a property that generalizes the notion of a subset of Euclidean space being closed and bounded. A topological space is sequentially compact if every infinite sequence of points sampled from the space has an infinite subsequence that converges to some point of the space. The term compact set is sometimes a synonym for compact space, but usually refers to a compact subspace of a topological space. Typical examples of compact spaces include spaces consisting not of geometrical points but of functions. The term compact was introduced into mathematics by Maurice Fréchet in 1904. In this unit, you will study connected spaces, connected components and disconnected spaces with examples. You will understand path connectedness and arc connectedness. You will describe locally connected space and their properties. Connected sets in the real line is also explained in this unit.

9.1 OBJECTIVES

After going through this unit you will be able to:

- Understand connected spaces, connected components and disconnected spaces

- Analyse path connectedness and arc connectedness
- Comprehend locally connected space and their properties
- Describe connected sets in the real line

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9.2 CONNECTED SPACES

In topology and related branches of mathematics, a connected space is a topological space that cannot be represented as the union of two or more disjoint nonempty open subsets.

Connectedness is one of the principal topological properties that is used to distinguish topological spaces. A stronger notion is that of a path connected space, which is a space where any two points can be joined by a path.

A subset of a topological space X is a connected set if it is a connected space when viewed as a subspace of X.

As an example of a space that is not connected, one can delete an infinite line from the plane. Other examples of disconnected spaces (that is, spaces that are not connected) include the plane with a closed annulus removed, as well as the union of two disjoint open disks in two dimensional Euclidean space.

Definition: A topological space X is said to be disconnected if it is the union of two disjoint nonempty open sets. Otherwise, X is said to be connected. A subset of a topological space is said to be connected if it is connected under its subspace topology.

For a topological space X the following conditions are equivalent:

1. X is connected.
2. X cannot be divided into two disjoint nonempty closed sets.
3. The only subset of X which are both open and closed (clopen sets) are X and the empty set.
4. The only subsets of X with empty boundary are X and the empty set.
5. X cannot be written as the union of two nonempty separated sets.
6. The only continuous functions from X to $\{0, 1\}$ are constant.

Connected Components

The maximal connected subsets of a nonempty topological space are called the connected components of the space. The components of any topological space X from a partition of X: they are disjoint, nonempty and their union is the whole space. Since we are insisting on calling the empty topological space connected, we need a special convention here: the empty space has no connected components.

Every component is a closed subset of the original space. It follows that, in the case where their number is finite, each component is also an open subset. However, if their number in infinite, this might not be the case; for example, the connected components of the set of the rational numbers are the one-point sets, which are not open.

Let Γ_x be a connected component of x in a topological space X, and Γ'_x be the intersection of all open-closed sets containing x (called quasi-component of x.) Then $\Gamma_x \subset \Gamma'_x$ where the equality holds if X is compact Hausdorff or locally connected.

Disconnected Spaces

A space in which all components are one-point sets is called totally disconnected. Related to this property, a space X is called totally separated if, for any two elements x and y of X, there exist disjoint open neighborhoods U of x and V of y such that X is the union of U and V. Clearly any totally separated space is totally disconnected, but the converse does not hold. For example, take two copies of the rational numbers Q and identify them at every point except zero. The resulting space, with the quotient topology, is totally disconnected. However, by considering the two copies of zero, is obvious that the space is not totally separated. In fact, it is not even Hausdorff, and the condition of being totally separated is strictly stronger than the condition of being Hausdorff.

The following are the examples:

- The closed interval [0, 2] in the standard subspace topology is connected; although it can, for example, be written as the union of [0,1) and [1, 2], the second set is not open in the aforementioned topology of [0, 2].
- The union of [0,1) and (1, 2] is disconnected; both of these intervals are open in the standard topological space $[0,1)\cup(1, 2]$.
- $(0,1)\cup\{3\}$ is disconnected.
- A convex set is connected; it is actually simply connected.
- A Euclidean plane excluding the origin, (0,0) is connected but is not simply connected. The three-dimensional Euclidean space without the origin is connected and even simply connected. In contrast, the one-dimensional Euclidean space without the origin is not connected.
- A Euclidean plane with a straight line removed is not connected since it consists of two half-planes.
- The space of real numbers with the usual topology is connected.
- Any topological vector space over a connected field is connected.

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- Every discrete topological space with at least two elements is disconnected, in fact such a space is totally disconnected. The simplest example is the discrete two-point space.
- The Cantor set is totally disconnected; since the set contains uncountably many points it has uncountably many components.
- If a space X is homotopy equivalent to a connected space, then X is itself connected.
- The topologist's sine curve is an example of a set that is connected but is neither path connected nor locally connected.
- The general linear group GL(n, **R**)(that is, the group of n-by-n real matrices) consists of two connected components: the one with matrices of positive determinant and the other of negative determinant. In particular, it is not connected. In contrast, GL(n,**C**) is connected. More generally, the set of invertible bounded operators on a (complex) Hilbert space is connected.
- The spectrum of a commutative local ring is connected. More generally, the spectrum of a commutative ring is connected if and only if it has no idempotent $\neq 0$, if and only if the ring is not a product of two rings in a nontrivial way.

Path Connectedness

A path from a point x to a point y in a topological space X is a continuous function f from the unit interval $[0,1]$ to X with $f(0) = x$ and $f(1) = y$. A path-component of X is an equivalence class of X under the equivalence relation which makes x equivalent to y if there is a path from x to y. The space X is said to be path-connected (or pathwise connected or 0-connected) if there is atmost one path-component, i.e., if there is a path joining any two points in X. Again, many others exclude the empty space.

Every path-connected space is connected. The converse is not always true: examples of connected spaces that are not path-connected include the extended long line L^* and the topologist's sine curve.

However, subsets of the real line **R** are connected if and only if they are path connected; these subsets are the intervals of **R**. Also, open subsets of $\mathbf{R}^n$ or $\mathbf{C}^n$ are connected if and only if they are path-connected. Additionally, connectedness and path-connectedness are the same for finite topological spaces.

Arc Connectedness

A space X is said to be arc-connected or arcwise connected if any two distinct points can be joined by an arc, that is a path f which is a homeomorphism between the unit interval $[0,1]$ and its image $f([0,1])$. It can be shown any Housdorff space which is path-connected is also arc-connected. An example of a space which is path-connected but not arc-connected is provided by adding a second copy $0'$ of 0 to the nonnegative real numbers $[0,\infty)$. One endows this set with a partial order by specifying that $0' < a$ for any positive number a, but leaving 0 and $0'$ incomparable. One then endows this set with the order topology, that is one takes the open intervals $(a, b) = \{x|a < x < \mathrm{b}\}$ and the half-open intervals $[0, a) = \{x|0 \leq x < a\}$, $[0', a) = \{x|0' \leq x < a\}$ as a base for the topology. The resulting space is a $\mathbf{T}_1$ space but not a Hausdorff space. Clearly 0 and $0'$ can be connected by a path but not an arc in this space.

Locally Connected Space

In topology and other branches of mathematics, a topological space X is locally connected if every point admits a neighbourhood basis consisting entirely of open connected sets.

Throughout the history of topology, connectedness and compactness have been two of the most widely studied topological properties. Indeed, the study of these properties even among subsets of Euclidean space, and the recognition of their independence from the particular form of the Euclidean metric, played a large role in clarifying the notion of a topological property and thus a topological space. However, whereas the structure of compact subsets of Euclidean space was understood quite early on via the Heine-Borel theorem, connected subsets of $\mathbf{R}^n$. (For $n > 1$) proved to be much more complicated. Indeed, while any compact Hausdorff space is locally compact, a connected a space and even a connected subset of the Euclidean plane need not be locally connected.

Although the basic point-set topology of manifolds is relatively simple, their algebraic topology is far more complex. From this modern perspective, the stronger property of local path connectedness turns out to be more important, for example, in order for a space to admit a universal cover it must be connected and locally path connected.

A space is locally connected if and only if for every open set U, the connected components of U are open. It follows that a continuous function from a locally connected space to a totally disconnected space must be locally constant. In fact the openness of components is so natural that one must be sure to keep in mind that it is not true in general: for instance Cantor space is totally disconnected but not discrete.

The following example will make the concept clear:

Let X be a topological space, and let x be a point of X.

We say that X is locally connected at x if for every open set V containing x there exists a connected, open set U with $x \in U \subset V$. The space X is said to be locally connected if it is locally connected at x for all x in X.

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By contrast, we say that X is weakly locally connected at x if for every open set V containing x (or connected im Kleinen at x) there exists a connected subset N of V such that x lies in the interior of N. An equivalent definition is: each open set V containing x contains an open neighborhood U of x such that any two points in U lie in some connected subset of V. The space X is said to be weakly locally connected if it is weakly locally connected at x for all x in X.

In other words, the only difference between the two definitions is that for local connectedness at x we require a neighbourhood base of open connected sets, whereas for weak local connectedness at x we require only a base of neighbourhoods of x.

Evidently a space which is locally connected at x is weakly locally connected at x. The converse does not hold. On the other hand, it is equally clear that a locally connected space is weakly locally connected and here it turns out that the converse does hold. A space which is weakly locally connected at all of its points is necessarily locally connected at all of its points.

We say that X is locally path connected at x if for every open set V containing x there exists a path connected, open set U with $x \in U \subset V$. The space X is said to be locally path connected at x for all x in X.

Since path connected spaces are connected, locally path connected spaces are locally connected.

The following examples will make the concept clear:

1. For any positive integer n, the Euclidean space $\mathbf{R}^n$ is connected and locally connected.
2. The subspace $[0,1]\cup[2,3]$ of real line $\mathbf{R}^1$ is locally connected but not connected.
3. The topologist's sine curve is a subspace of the Euclidean plane which is connected, but not locally connected.
4. The space $\mathbf{Q}$ of rational numbers endowed with the standard Euclidean topology, is neither connected nor locally connected.
5. The comb space is path connected but not locally path connected.
6. Let X be a countably infinite set endowed with the cofinite topology. Then X is locally connected (indeed hyperconnected) but not locally path connected.

Properties

The following are the properties of connectedness:

1. Local connectedness is, by definition, a local property of topological spaces, i.e., a topological property P such that a space X possesses property P if

and only if each point x in X admits a neighbourhood base of sets which have property P. Accordingly, all the metaproperties held by a local property hold for local connectedness.

2. A space is locally connected if and only if it admits a base of connected subsets.
3. The disjoint union $\coprod_i X_i$ of a family $\{X_i\}$ of spaces is locally connected if and only if each X_i is locally connected. In particular, since a single point is certainly locally connected, it follows that any discrete space is locally connected. On the other hand, a discrete space is totally disconnected, so is connected only if it has at most one point.
4. Conversely, a totally disconnected space is locally connected if and only if it is discrete. This can be used to explain the aforementioned fact that the rational numbers are not locally connected.

9.3 CONNECTED SETS IN THE REAL LINE

The notion of connectedness is of fundamental importance in analysis. Before giving a formal definition of connectedness in a metric space, we introduce the notion of subspace.

Let (X, d) be a metric space and let A be a subset of X. Let d^* denote restriction of d *to* $A \times A$, that is,

$$d^* (x, y) = d (x, y)$$

Here x, y are points of A. Then d^* is a metric for A called the induced metric and the set A with metric d^* is called subspace of X.

Thus, a subset A of X equipped with the induced metric is a metric space in its own right and neighbourhoods, open sets and closed sets are defined as a metric space. But an open set (closed set) of A need not be open (closed) when regarded as a subset of X.

For example, if we regard the closed interval [0, 1] as a sub-space of **R**, then the semi-open interval [0, 1[is open in [0, 1] but not in **R**. In fact, if A is a subspace of X and $B \subset A$, then

(i) B is open in if there exists a set **G** open in X such that

$$B = \mathbf{G} \cap A.$$

(ii) B is closed in A if there exists a set H closed in X such that

$$B = H \cap A.$$

Note that the phrase 'B is open in A' means that B is open relative to the induced metric on A. Also 'B is open in X' means that B is open with respect to the metric on X.

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Notes: (i) If $A \subset X$ is open, then $B \subset A$ is open in A if it is open in X.

(ii) If $A \subset X$ is closed, then $B \subset A$ is closed in A if it is closed in X.

Another definition is subset A of a metric space X is said to be disconnected if it is the union of two nonempty disjoint sets both open in A such that,

$$C \cap \mathrm{D} = \varnothing \text{ and } C \cup D = A.$$

It follows from the preceding definition that a subset of a metric space X is disconnected if it is the union of two non-empty disjoint sets both closed in A.

We call $C \cap D$ the separation (or disconnection) of A.

It follows at once from definition that every singleton set is a connected set.

Theorem 1: R is connected.

Proof: Suppose, if possible, R is disconnected. Then there exist two non-empty, disjoint, closed sets A and B such that $\mathrm{R} = A \cup B$. Since A, B are non-empty, we can find $a_1 \in A$ and $b_1 \in B$. Since $A \cup B = \varnothing$, $a_1 \neq b_1$, and so either $a_1 > b_1$ or $a_1 < b_1$. Suppose $a_1, < b_1$. Let $I_1 =]a_1, b_1]$ so that I_1 is a closed interval. Bisect I_1 and observe that its mid-point $\dfrac{a_1 + b_1}{2}$ must belong either to A or to B but not both since A and B are disjoint. It follows that one of the two halves must have its left end point in A and its right end point in B. We denote this interval by $I_2 = [a_2, b_2]$. We bisect I_2 and proceed as before. We continue this process ad infinitum. Evidently $I_1 \supset I_2 \supset I_3 \supset$... Thus we obtain a nested sequence $<I_n>$ of closed intervals such that their length $|I_n| \to 0$ as $n \to \infty$. Hence, by the nested interval theorem, there exists a unique point c which belongs to every I_n, that is

$$c \in \cap \{In : n \in \mathrm{N}\}.$$

It is easy to see that c is a limit point of both A and B. For if $\mathrm{V}(c, \in) =]c - \in, c + \in[$ is any $\in$-*nhd* of c, we can find a positive integer m_0 so large that $I_n \subset N(c, \in)$ for all $n \geq m_0$ and consequently $N(c, \in)$ contains infinite number of points of both A and B. Since A and B are closed, $c \in A$ and $c \in B$ which is a contradiction since $A \cap B = \varnothing$.

Hence, **R** must be connected.

Theorem 2: A sebset A of **R** is connected if and only if it is an interval.

Proof: The 'only if' part. Let A be connected and suppose if possible, A is not an interval. Then there exist real numbers a, p, b with $a < p < b$ such that $a, b \in A$ but $p \notin A$. Let, $\mathbf{G} =]-\infty, p[$ and $\mathbf{H} =]p, x[$.

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Then **G**, **H** are disjoint, nonempty open sets in **R**. They are nonempty since $a \in \mathbf{G}$ and $b \in \mathbf{H}$. Let

$$\mathbf{C} = A \cap \mathbf{G} \text{ and } \mathbf{D} = A \cap \mathbf{H}.$$

Then, **C** and **D** are open in A. Further $a \in \mathbf{C}$ and b and **D** so that they are nonempty. Also

$$\mathbf{C} \subset \mathbf{G}, \mathbf{D} \subset \mathbf{H} \text{ and } \mathbf{G} \cap \mathbf{H} = \varnothing \Rightarrow \mathbf{C} \cap \mathbf{D} = \varnothing,$$

$$\text{and } \mathbf{C} \cup \mathbf{D} = (A \cap \mathbf{G}) \cup (\mathbf{B} \cap \mathbf{H})$$

$$= A \cap (\mathbf{G} \cup \mathbf{H}) = A \cap (\mathbf{R} - \{p\}) = A$$

$$[\because p \notin A \Rightarrow A \subset \mathbf{R} - \{p\}]$$

Hence, $\mathbf{C} \cup \mathbf{D}$ is a separation of A and consequently A is disconnected which is contradiction. Hence, A must be an interval.

The 'if' part. The proof of this part is exactly on the same lines and is left as an exercise.

Check Your Progress

1. What do you mean by a connected space?
2. What is meant by connectedness?
3. When is a topological set space said to be connected?
4. What are connected components?
5. What is path connectedness?
6. Define the term arc connectedness.

9.4 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1. A connected space is a topological space that cannot be represented as the union of two or more disjoint nonempty open subsets.
2. Connectedness is one of the principal topological properties that is used to distinguish topological spaces. A stronger notion is that of a path connected space, which is a space where any two points can be joined by a path.
3. A topological space X is said to be disconnected if it is the union of two disjoint nonempty open sets. Otherwise, X is said to be connected. A subset of a topological space is said to be connected if it is connected under its subspace topology.
4. The maximal connected subsets of a nonempty topological space are called the connected components of the space.

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5. The space X is said to be path-connected (or pathwise connected or 0-connected) if there is atmost one path-component, i.e., if there is a path joining any two points in X. Again, many others exclude the empty space.
6. A space X is said to be arc-connected or arcwise connected if any two distinct points can be joined by an arc, that is a path f which is a homeomorphism between the unit interval $[0,1]$ and its image $f([0,1])$.

9.5 SUMMARY

- In topology and related branches of mathematics, a connected space is a topological space that cannot be represented as the union of two or more disjoint nonempty open subsets.
- Connectedness is one of the principal topological properties that is used to distinguish topological spaces. A stronger notion is that of a path connected space, which is a space where any two points can be joined by a path.
- A subset of a topological space X is a connected set if it is a connected space when viewed as a subspace of X.
- As an example of a space that is not connected, one can delete an infinite line from the plane.
- A topological space X is said to be disconnected if it is the union of two disjoint nonempty open sets. Otherwise, X is said to be connected. A subset of a topological space is said to be connected if it is connected under its subspace topology.
- The maximal connected subsets of a nonempty topological space are called the connected components of the space.
- Let Γ_x be a connected component of x in a topological space X, and Γ'_x be the intersection of all open-closed sets containing x (called quasi-component of x.) Then $\Gamma_x \subset \Gamma'_x$ where the equality holds if X is compact Hausdorff or locally connected.
- A space in which all components are one-point sets is called totally disconnected.
- Any topological vector space over a connected field is connected.
- Every discrete topological space with at least two elements is disconnected, in fact such a space is totally disconnected. The simplest example is the discrete two-point space.
- The Cantor set is totally disconnected; since the set contains uncountably many points it has uncountably many components.

- The space X is said to be path-connected (or pathwise connected or 0-connected) if there is atmost one path-component, i.e., if there is a path joining any two points in X. Again, many others exclude the empty space.
- A space X is said to be arc-connected or arcwise connected if any two distinct points can be joined by an arc, that is a path f which is a homeomorphism between the unit interval [0,1] and its image $f([0,1])$.
- In topology and other branches of mathematics, a topological space X is locally connected if every point admits a neighbourhood basis consisting entirely of open connected sets.
- A space is locally connected if and only if for every open set U, the connected components of U are open.
- for local connectedness at x we require a neighbourhood base of open connected sets, whereas for weak local connectedness at x we require only a base of neighbourhoods of x.
- A space which is weakly locally connected at all of its points is necessarily locally connected at all of its points.
- We say that X is locally path connected at x if for every open set V containing x there exists a path connected, open set U with $x \in U \subset V$. The space X is said to be locally path connected at x for all x in X.
- a subset A of X equipped with the induced metric is a metric space in its own right and neighbourhoods, open sets and closed sets are defined as metric space. But an open set (closed set) of A need not be open (closed) when regarded as a subset of X.
- subset A of a metric space X is said to be disconnected if it is the union of two nonempty disjoint sets both open in A such that,

  $C \cap \mathrm{D} = \varnothing$ and $C \cup D = A$.

9.6 KEY WORDS

- **Connected space:** The maximal connected subsets of a nonempty topological space are called the connected components of the space.
- **Metric space:** A space in which all components are one-point sets is called totally disconnected.
- **Path connectedness:** The space X is said to be path-connected (or pathwise connected or 0-connected) if there is atmost one path-component, i.e., if there is a path joining any two points in X. Again, many others exclude the empty space.

- **Arc connectedness:** A space X is said to be arc-connected or arcwise connected if any two distinct points can be joined by an arc, that is a path f which is a homeomorphism between the unit interval [0,1] and its image $f([0,1])$.
- **Locally connected space:** In topology and other branches of mathematics, a topological space X is locally connected if every point admits a neighbourhood basis consisting entirely of open connected sets.

9.7 SELF ASSESSMENT QUESTIONS AND EXERCISES

Short Answer Questions

1. Write a short notes on the followings:
   (*i*) Connected spaces
   (*ii*) Disconnected spaces
   (*iii*) Connected components
   (*iv*) Path connectedness
2. Differentiate between arc connectedness and locally connected space.
3. Write properties of connectedness.

Long Answer Questions

1. Discuss connected spaces with suitable examples.
2. Give a detailed of the connected sets in the real line.

9.8 FURTHER READINGS

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UNIT 10 LOCAL CONNECTEDNESS AND COMPACT SPACES

Structure

10.0 INTRODUCTION

Connectedness and compactness have been two of the most widely studied topological properties throughout the history of topology. If a space is locally connected at each of its points, we call the space locally connected. In general topology, compactness is a property that generalizes the notion of a subset of Euclidean space being closed and bounded. A topological space is compact if every open cover of X has a finite sub cover. In this unit, you will study local connectedness and compact spaces. You will understand compactness, nets and para-compact spaces applying some mathematical theorems.

10.1 OBJECTIVES

After going through this unit, you will be able to:

- Understand local connectedness and compact spaces
- Analyse compactness and nets
- Define paracompact spaces

10.2 LOCAL CONNECTEDNESS AND COMPACT SPACES

Connectedness and compactness have been two of the most widely studied topological properties throughout the history of topology. Certainly, the study of these properties even among subsets of Euclidean space and the recognition of their independence from the particular form of the Euclidean metric played a large

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role in clarifying the notion of a topological property and thus a topological space. However, whereas the structure of compact subsets of Euclidean space was understood quite early on via the Heine-Borel theorem, connected subsets of $\mathbf{R}^n$ (for $n > 1$) proved to be much more complicated. Indeed, while any compact Hausdorff space is locally compact, a connected space – and even a connected subset of the Euclidean plane – need not be locally connected.

In the latter part of the twentieth century, research trends shifted to more intense study of spaces like manifolds which are locally well understood being locally homeomorphic to Euclidean space but have complicated global behavior. This means that although the basic point-set topology of manifolds is relatively simple since manifolds are essentially metrizable according to most definitions of the concept, their algebraic topology is far more complex. From this modern viewpoint, the stronger property of local path connectedness turns out to be more important, for example, in order for a space to admit a universal cover it must be connected and locally path connected. A space is said to be locally connected iff for every open set U, the connected components of U in the subspace topology are open. A continuous function from a locally connected space to a totally disconnected space must be locally constant.

Theorem 1: $\prod_{\lambda} X_\lambda$ is locally connected if and only if each space X_λ is locally connected and all but a finite number are connected.

Proof: Suppose $\prod_{\lambda} X_\lambda$ is locally connected, and let $x_\beta \in X_\beta$ be contained in some open set Y_β. Choose some point $z = <z_\lambda>$ with $z_\beta = X_\beta$ and we have z belonging to the open set $\pi_\beta^{-1}(Y_\beta)$. By local connectedness, there must exist a connected open set G containing z and contained in $\pi_\beta^{-1}(Y_\beta)$.Taking the β-th projection, we see that $z_\beta = x_\beta$ is contained in the connected open set $\pi_\beta(G)$ which is itself contained in Y_β and so X_β is locally connected. Further, if z is any point of the product space, it must be contained in some connected open set G. By definition, $z \in \underset{\lambda}{\pi} Y_\lambda \subseteq G$ where Y_λ is open in X_λ for all λ and $Y_\lambda = X_\lambda$ for all but a certain finite number of values of λ. But then the projections of G are connected and are equal to X_λ, except for that finite number of values of λ.

Now suppose that X_λ is locally connected for old λ and connected for $\lambda \neq \beta_1, \beta_2, ..., \beta_n$. Let $X = <x_\lambda>_\lambda$ be an arbitrary point of $\underset{\lambda}{\pi} Y_\lambda$ where Y_λ is open in X_λ for all λ and $Y_\lambda = X_\lambda$ for $\lambda \neq \beta_1{*}, \beta_2{*}, ..., \beta_k{*}$. Since $x_\lambda \in Y_\lambda$ for all λ and Y_λ is locally connected, there is a connected open set G_λ in X_λ such that $x_\lambda \in G_\lambda \subseteq Y_\lambda$.

Consider the subset $\underset{\lambda}{\pi} Z_\lambda$ where $Z_\lambda = G_\lambda$ if $\lambda = \beta_1, \beta_2, ..., \beta_n, \beta_1*,...., \beta_k*$ and $Z_\lambda = X_\lambda$ otherwise. But by the result, $\prod_\lambda X_\lambda$ is connected if and only if each X_λ is connected. This set is connected. Hence we have formed a connected open set containing X and contained in $\prod_\lambda Y_\lambda$.

10.3 COMPACTNESS AND NETS

Definition: We say that a net $\{x_\lambda\}$ has $x \in X$ as a cluster point if and only if for each neighbourhood U of x and for each $\lambda_0 \in \Lambda$ there exist some $\lambda \geq \lambda_0$ such that $x_\lambda \in U$. In this case we say that $\{x_\lambda\}$ is cofinally (or frequently) in each neighbourhood of x.

Theorem 2: A net $\{x_\lambda\}$ has $y \in X$ as a cluster point if and only if it has a sub net, which converges to y.

Proof: Let y be a cluster point of $\{x_\lambda\}$. Define $M := \{(\lambda, U) : \lambda \in \Lambda, U$ a neighbourhood of y such that $x_\lambda \in U\}$ and order M as follows: $(\lambda_1, U_1) \leq (\lambda_2, U_2)$ if and only if $\lambda_1 \leq \lambda_2$ and $U_2 \subseteq U_1$. This is easily verified to be a direction on M. Define $\varphi : M \to \Lambda$ by $\varphi(\lambda, U) = \lambda$. Then φ is increasing and cofinal in Λ, so φ defines a subnet of $\{x_\lambda\}$. Let U_0 be any neighbourhood of y and find $\lambda_0 \in \Lambda$ such that $x_{\lambda 0} \in U_0$. Then $(\lambda_0, U_0) \in M$, and moreover, $(\lambda, U) \geq (\lambda_0, U_0)$ implies $U \subseteq U_0$, so that $x_\lambda \in U \subseteq U_0$. It follows that the subnet defined by φ converges to y.

Suppose $\varphi : M \to \Lambda$ defines a subnet of $\{x_\lambda\}$ which converges to y. Then for each neighbourhood U of y, there is some u_U in M such that $u \geq u_U$ implies $x_{\varphi(u)} \in U$. Suppose a neighbourhood U of y and a point $\lambda_0 \in \Lambda$ are given. Since $\varphi(M)$ is cofinal in Λ, there is some $u_0 \in M$ such that $\varphi(u_0) \geq \lambda_0$. But there is also some $u_U \in M$ such that $u \geq u_U$ implies $x_{\varphi(u)} \in U$. Pick $u* \geq u_0$ and $u* \geq u_U$. Then $\varphi(u*) = \lambda* \geq \lambda_0$, since $\varphi(u*) \geq \varphi(u_0)$, and $x_{\lambda*} = x_\varphi(u*) \in U$, since $u* \geq u_U$. Thus for any neighbourhood U of y and any $\lambda_0 \in \Lambda$, there is some $\lambda* \geq \lambda$ with $x_{\lambda*} \in U$. It follows that y is a cluster point of $\{x_\lambda\}$.

Theorem 3: A topological space X is compact if and only if every net on X has a convergent subnet on X.

Proof: Assume that X is compact and suppose that we have a net $\{x_\lambda\}$ that does not has any convergent subnet. Hence, using the previous Theorem 4.23, the net $\{x_\lambda\}$ does not has cluster points. This means that for each $x \in X$ we can find a neighbourhood U_x of x and an index λ_x such that $x_\lambda \notin U_x$ for every $\lambda \geq \lambda_x$. Since X is compact then there exist $x_1, x_2, \ldots, x_n \in X$ such that $X = \bigcup_{i=1} U_{xi}$. Take any $\lambda \geq \lambda_{x1}, \lambda_{x2}, ..., \lambda_{xn}$. Then $x_\lambda \notin X$ which is contradiction.

Assume that every net on X has a convergent subnet on X. We will show that X is compact. To this end take a family $\mathcal{F} = \{F_i : i \in \mathbf{I}\}$ of closed subnets of X with the finite intersection property, that is $F_{i1} \cap F_{i2} \cap ... \cap F_{in} \neq \phi$ for every $\{i_1, i_2, ..., i_n\} \subseteq \mathbf{I}$. We will show that $\bigcap_{i \in I} F_i \neq \phi$. Define a net as follows: Let $\Lambda = \{\{i_1, i_2, ..., i_n\} : i_1, i_2, ..., i_n \in \mathbf{I} \text{ and } n \in \mathbf{N}\}$.

And order Λ as follows: $\lambda_1 = \{i_1, i_2, ..., i_k\} \leq \lambda_2 = \{j_1, j_2, ..., j_n\}$ if and only if $\{i_1, i_2, ..., i_k\} \subseteq \{j_1, j_2, ..., j_n\}$. This is easily verified to be a direction on Λ. Since the family F has the finite intersection property then for every $\lambda = \{i_1, i_2, ..., i_n\} \in \Lambda$ we can find $x_\lambda \in F_{i1} \cap F_{i2} \cap, ..., \cap F_{i_n}$. Using our hypothesis, the net $\{x_\lambda\}$ has a convergent subnet, let say $\{x_{\lambda m}\}$. That is, there exists $x \in X$ such that $x_{\lambda m} \to x$. We will show that $x \in F_i$ for all $i \in \mathbf{I}$. Fix some F_i. Hence,there exists m_0 such that $\lambda_{m0} \geq \{i\}$. Thus, for every $\lambda_m = \{i_1, i_2, ..., i_n, i\} \geq \lambda_{m0} \geq \{i\}$ we have that $x_{\lambda m} \in F_{i1} \cap F_{i2} \cap ... \cap F_{in} \cap F_i \subseteq F_i$. Since $x_{\lambda m} \to x$ and F_i is closed then $x \in F_i$. This finishes the proof of the theorem.

10.4 PARACOMPACT SPACES

In mathematics, a *paracompact space* is a topological space in which every open cover admits a locally finite open refinement. Paracompact spaces are sometimes also required to be Hausdorff. Paracompact spaces were introduced by Dieudonné (1944).

Let X be a topological space.

Definition: The space X is locally compact if each $x \in X$ admits a compact neighbourhood N.

If X is locally compact and Hausdorff, then all compact sets in X are closed and hence if N is a compact neighbourhood of x then N contains the closure the open int (N) around x. Hence, in such cases every point $x \in X$ lies in an open cover whose closure is compact. Much more can be said about the local structure of locally compact Hausdorff spaces, though it requires some serious theorems in topology (such as Urysohn's lemma).

Lemma: If X is a locally compact Hausdorff space that is second countable, then it admits a countable base of opens $\{U_n\}$ with compact closure.

Proof: Let $\{V_n\}$ be a countable base of opens. For each $x \in X$ there exists an open U_x around x with compact closure, yet some $V_{n(x)}$ contains x and is contained in U_x. The closure of $V_{n(x)}$ is a closed subset of the compact $\overline{U_x}$, and so $\overline{V}_{n(x)}$ is also compact. Thus, the V_n's with compact closure are a countable base of opens with compact closure.

Definition: An open covering $\{U_i\}$ of X refines an open covering $\{V_j\}$ of X if each U_i is contained in some V_j.

A simple example of a refinement is a subcover, but refinements allow much great flexibility: none of the U_i's needs to be a V_j. For example, the covering of a metric space by open balls of radius 1 is refined by the covering by open balls of radius $\frac{1}{2}$.

Definition: An open covering $\{U_i\}$ of X is locally finite if every $x \in X$ admits a neighbourhood N such that $N \cap U_i$ is empty for all but finitely many i.

For example, the covering of **R** by open intervals $(n–1, n+1)$ for $n \in \mathbf{Z}$ is locally finite, whereas the covering of $(-1,1)$ by intervals $(-1/n, 1/n)$ (for $n \geq 1$) barely fails to be locally finite: there is a problem at the origin (but nowhere else).

Definition: A topological space X is paracompact if every open covering admits a locally finite refinement. It is traditional to also require paracompact spaces to be Hausdorff, as paracompactness is never used away from the Hausdorff setting, in contrast with compactness though many mathematicians implicity require compact spaces to be Hausdorff too and they reserve a separate word (quasi-compact) for compactness without the assumption of the Hausdorff condition.

Obviously any compact space is paracompact. Also, an arbitrary disjoint union $\cup X_i$ of paracompact spaces (given the topology wherein an open set is one that meets each X_i is an open subset) is again paracompact. Note that it is not the case that open covers of a paracompact space admit locally finite sub covers, but rather just locally finite refinements. Indeed, we saw at the outset that $\mathbf{R}^n$ is paracompact, but even in the real line, there exist open covers with no locally finite sub cover: let $U_n = (-\infty, n)$ for $n \geq 1$. All U_n's contain $(-\infty, 0)$, and any sub collection of U_{n_i} 's that covers $\mathbf{R}$ has to be infinite since each U_n is bounded on the right. Thus, no subcover can be locally finite near a negative number.

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In general, paracompactness is a slightly tricky property: there are counter examples that show that an open subset of a paracompact Hausdorff space need not be paracompact. Thus, to prove that an open subset of $\mathbf{R}^n$ is paracompact we will have to use special features of $\mathbf{R}^n$. However, just as closed subsets of compact sets are compact, closed subsets of paracompact spaces are paracompact; the argument is virtually the same as in the compact case (extend covers by using the complement of the closed set). It is non-trivial theorem in topology that any metric space is paracompact.

Theorem 4: Any second countable Hausdorff space X that is locally compact is paracompact.

Proof: Let $\{V_n\}$ be a countable base of opens in X. Let $\{U_i\}$ be an open cover of X for which we seek a locally finite refinement. Each $x \in X$ lies in some U_i and so there exists a $V_{n(x)}$ containing x with $V_{n(x)} \subseteq U_i$. The $V_{n(x)}$'s therefore constitute a refinement of $\{U_i\}$ that is countable. Since the property of one open covering refining another is transitive, we therefore lose no generality by seeking locally finite refinements of countable covers. By Lemma stated above, we can assume that all $\overline{V}_n$ are compact. Hence, we can restrict our attention to countable covers by opens U_n for which $\overline{U_n}$ is compact. Since closure commutes with finite unions, by replacing U_n with $\bigcup_{j \leq n} U_j$ we preserve the compactness condition (as a finite union of compact subsets is compact) and so we can assume that $\{U_n\}$ is an increasing collection of opens with compact closure (with $n \geq 0$). Since $\overline{U_n}$ is compact yet is covered by the open U_i's, for sufficiently large N we have $\overline{U_n} \subseteq U_N$. If we recursively replace U_{n+1} with such a U_N for each n, then we can arrange that $\overline{U_n} \subseteq U_{n+1}$ for each n. Let $K_0 = \overline{U_0}$ and for $n \geq 1$ let $K_n = \overline{U_n} - U_{n-1} = \overline{U_n} \cap (X - U_{n-1})$, so K_n is compact for every n (as it is closed in the compact $\overline{U_n}$) but for any fixed N we see that U_N is disjoint from K_n for all $n \geq N$.

Now we have a situation similar to the concentric shells in our proof of paracompactness of $\mathbf{R}^n$, and so we can carry over the argument from Euclidean spaces as follows. We seek a locally finite refinement of $\{U_n\}$. For $n \geq 2$, the open set $W_n = U_{n+1} - \overline{U}_{n-2}$ contains K_n, so for each $x \in K_n$ there exists some $V_m \subseteq W_n$ around x. There are finitely many such V_m's that actually cover the compact K_n, and the collection of V_m's that arise in this way as we vary $n \geq 2$ is clearly a locally finite collection of opens in X whose union contains $X - \overline{U_0}$. Throwing infinitely many V_m's contained in U_1 that cover the compact $\overline{U_0}$ thereby gives an open cover of X that refines $\{U_i\}$ and is locally finite.

Corollary: Let X be a Hausdorff topological premanifold. The following properties of X are equivalent: its connected components are countable unions of compact sets, its connected components are second countable, and it is paracompact.

Proof: If $\{U, V\}$ is a separation of X and X is paracompact then it is clear that both U and V are paracompact. Hence, since the connected components of X are open, X is paracompact if and only if its connected components are paracompact. We may therefore restrict our attention to connected X. For such X, we claim that it is equivalent to require that X be a countable union of compact sets, that X be second countable, and that X be paracompact. By the preceding theorem, if X is second countable then it is paracompact. Since X is connected, Hausdorff, and locally compact, if it is paracompact then it is a countable union of compacts. Hence, to complete the cycle of implications it remains to check that if X is a countable union of compacts then it is second countable. Let $\{K_n\}$ be a countable collection of compacts that cover X, so if $\{U_i\}$ is a covering of X by open sets homeomorphic to an open set in a Euclidean space we may find finitely many U_i's that cover X. Since each U_i is certainly second countable (being, open in a Euclidean space), a countable base of opens for X is given by the union of countable bases of opens for each of the U_i's. Hence, X is second countable.

Check Your Progress

1. What is a locally connected space?
2. When a net does has a cluster point?
3. When is a topological space said to be compact?
4. What is a paracompact space?

10.5 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1. A space is said to be locally connected iff for every open set U, the connected components of U in the subspace topology are open. A continuous function from a locally connected space to a totally disconnected space must be locally constant.

NOTES

2. We say that a net $\{x_\lambda\}$ has $x \in X$ as a cluster point if and only if for each neighbourhood U of x and for each $\lambda_0 \in \Lambda$ there exist some $\lambda \geq \lambda_0$ such that $x_\lambda \in U$.
3. A topological space X is compact if and only if every net on X has a convergent subnet on X.
4. A paracompact space is a topological space in which every open cover admits a locally finite open refinement.

10.6 SUMMARY

In this unit, you have learned that:

- While any compact Hausdorff space is locally compact, a connected space – and even a connected subset of the Euclidean plane – need not be locally connected.
- $\prod_\lambda X_\lambda$ is locally connected if and only if each space X_λ is locally connected and all but a finite number are connected.
- By local connectedness, there must exist a connected open set G containing z and contained in $\pi_\beta^{-1}(Y_\beta)$.
- We say that a net $\{x_\lambda\}$ has $x \in X$ as a cluster point if and only if for each neighbourhood U of x and for each $\lambda_0 \in \Lambda$ there exist some $\lambda \geq \lambda_0$ such that $x_\lambda \in U$.
- A net $\{x_\lambda\}$ has $y \in X$ as a cluster point if and only if it has a sub net, which converges to y.
- A topological space X is compact if and only if every net on X has a convergent subnet on X.
- A paracompact space is a topological space in which every open cover admits a locally finite open refinement.
- The space X is locally compact if each $x \in X$ admits a compact neighbourhood N.
- If X is locally compact and Hausdorff, then all compact sets in X are closed and hence if N is a compact neighbourhood of x then N contains the closure the open int (N) around x.

- If X is a locally compact Hausdorff space that is second countable, then it admits a countable base of opens $\{U_n\}$ with compact closure.
- An open covering $\{U_i\}$ of X refines an open covering $\{V_j\}$ of X if each U_i is contained in some V_j.
- An open covering $\{U_i\}$ of X is locally finite if every $x \in X$ admits a neighbourhood N such that $N \cap U_i$ is empty for all but finitely many i.
- A topological space X is paracompact if every open covering admits a locally finite refinement.
- Any second countable Hausdorff space X that is locally compact is paracompact.

10.7 KEY WORDS

- **Local connectedness:** If a space is locally connected at each of its points, we call the space locally connected.
- **Compact space:** A topological space is compact if every open cover of X has a finite sub cover.
- **Paracompact space:** A paracompact space is a topological space in which every open cover admits a locally finite open refinement.
- **Net:** In general topology a net is a generalization of the notion of a sequence.

10.8 SELF ASSESSMENT QUESTIONS AND EXERCISES

Short Answer Questions

1. Describe local connectedness briefly.
2. Show that a topological space X is compact if and only if every net on X has a convergent subnet on X
3. Prove that if X is a locally compact Hausdorff space that is second countable, then it admits a countable base of opens with compact closure.
4. Show that any second countable Hausdorff space X that is locally compact is paracompact.

Long Answer Questions

1. Discuss compactness and nets.
2. Give a detailed account of limit point compactness.

NOTES

10.8 FURTHER READINGS

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UNIT 11 COMPACT SET AND LIMIT POINT COMPACTNESS

Structure

11.0 INTRODUCTION

A topological space X is said to be limit point compact if every infinite subset of X has a limit point in X. A topological space is locally compact if each point has a relatively compact neighbourhood. Continuous image of a compact space is compact. Local compactness is not preserved by continuous surjections. In this unit you will study compact set in the real line. You will prove that every compact subset of a metric space is closed and closed subsets of compact sets are compact. You will discuss sequentially and countably compact sets to understand that a compact metric space is sequentially compact. You will describe Bolzano-Weierstrass property and sequential compactness, completeness and finite intersection property. You will discuss continuous functions and compact sets, characterisation of continuous functions using preimages and general properties of continuous functions. Limit point compactness and local compactness is also explained in this unit

11.1 OBJECTIVES

After going through this unit, you will be able to:

- Discuss compact set in the real line, sequentially and countably compact sets
- Describe Bolzano-Weierstrass property and sequential compactness, completeness and finite intersection property

- Explain continuous functions and compact sets, characterisation of continuous functions
- Interpret limit point compactness and local compactness

NOTES

11.2 COMPACT SET IN THE REAL LINE

By an open cover of a subset A of a metric space X, we mean a collection $C = \{G_\lambda : \lambda \in 1\}$ of open subsets of X such that $A \subset \cup \{G_\lambda : \lambda \in A\}$. We then say that C covers A.

In particular, C is said to be an open cover of the metric space X if $X = \cup \{G_\lambda : \lambda \in A\}$.

By a subcover of an open cover C of A, we mean a subcollection C' *of* C' such that C' covers A.

An open cover of A is said to be finite if it consists of finite number of open sets.

Another definition is, A subset of a metric space X is said to be compact if every open cover of A has a finite subcover, that is, if for every collection $\{G_\lambda : \lambda \in A\}$ of open sets for which

$$A \subset \cup \{G_\lambda : \lambda \in A\},$$

there exist finitely many sets $G_{\lambda 1},...,G_{\lambda n}$ among the G_λ's such that $A \subset G_{\lambda_1} \cup ... \cup G_{\lambda_n}$.

In particular, the metric space X is said to be compact if for every collection $\{G_\lambda : \lambda \in A;$ of open sets for which

$$X = \cup\{G_\lambda : \lambda \in A\},$$

there exist finitely many sets $G_{\lambda_1},...,G_{\lambda_n}$ among the G_λ's such that $X = G_{\lambda_1} \cup ... \cup G_{\lambda_n}$

Theorem 1: Let Y be a subspace of a metric space X and let $A \subset Y$. Then A is compact relative to X if and only if A is compact relative Y.

Proof: Let A be compact relative to X and let $\{V_\lambda, \lambda \in A\}$ be a collection of sets, open relative to Y, which covers A so that $A \subset \cup\{V_\lambda : \lambda \in A\}$. Then there exists G_λ, open relative to X, such that $V_\lambda = Y \cap G_\lambda$ for every $\lambda \in A$. It then follows that

$$A \subset \cup\{G_\lambda : \lambda \in A\},$$

that is, $\{G_\lambda : \lambda \in A\}$ is an open cover of A relative to X. Since A is compact relative to X, there exist finitely many indices $\lambda_1,...,\lambda_n$ such that

$$A \subset G_{\lambda_1} \cup, ..., \cup G_{\lambda_n}.$$

Since $A \subset Y$, we have $A = Y \cap A$.

Hence $A \subset Y \cap \{G_{\lambda_1} \cup, ..., \cup G_{\lambda_n}\} = (Y \cap G_{\lambda_1}) \cup ... \cup (Y \cap G_{\lambda_n})$

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[Distributive law]

Since $Y \cap G_{\lambda_i} = A_{\lambda_i} \{i = 1, 2, ..., n\}$, we obtain

$$A \subset A_{\lambda_1} \cup, ..., \cup A_{\lambda_n}. \quad .....(1)$$

This shows that A is compact relative to Y.

Conversely, let A be compact relative to Y and let $\{G_\lambda : \lambda \in \Lambda\}$ be a collection of open subsets of X which cover A so that

$$A \subset \cup \{G_\lambda : \lambda \in \Lambda\} \quad .....(2)$$

Since $A \subset Y$, Equation (2.2) implies that

$$A \subset Y \cap [\cup \{G_\lambda : \lambda \in \Lambda\} = \cup \{Y \cap G_\lambda : \lambda \in \Lambda\} \quad \text{[Distributive law]}$$

Since $Y \cap G_\lambda$ is open relative to Y, the collection

$$\{Y \cap G_\lambda : \lambda \in \Lambda\}$$

is an open cover of A relative to Y. Since A is compact relative to Y, we must have

$$A \subset (Y \cap G_{\lambda 1}) \cup ..... \cup (Y \cap G_{\lambda n}) \quad .....(3)$$

for some choice of finitely many indices $\lambda_1, ..., \lambda_n$. But (Equation 3) implies that

$$A \subset G_{\lambda 1} \cup ..... \cup G_{\lambda n}.$$

It follows that A is compact relative to X.

Theorem 2: Every compact subset of a metric space is closed.

Proof: Let A be a compact subset of a metric space X. We shall prove that $X - A$ is an open subset of X. Let $p \in X - A$.

For each $q \in A$, let $N_q(p)$ and $M(q)$ be neighbourhoods of p and q, respectively, of radius less than $\frac{1}{2} d\ (p, q)$ so that

$$N_q(p) \cap M(q) = \varnothing$$

Then the collection

$$\{M(q) : q \in A\}$$

is an open cover of A. (Recall that neighbourhoods are open sets). Since A is compact, there are finitely many points $q_1, ..., q_n$ in A such that

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$$A \subset M(q_1) \cup, ..., \cup N(q_n) \quad ......(4)$$

If $N = N_{q_1}(p) \cap, ..., \cap N_{q_n}(p)$ then N is a neighbourhood of p. Now

$N \subset N_{q_1}(p)$ and $M(q_i) \cap N_{qi}(p) = \varnothing$ for $i = 1,...,n$

$\Rightarrow M(q_i) \cap N = \varnothing$ for $i = 1,...,n$

$\Rightarrow \cup \{M(q_i) \cap N : i = 1,...,n = \varnothing$

$\Rightarrow [\cup \{M(q_i) : i = 1,...,n\}] \cap N = \varnothing$

[Distributive law]

$\Rightarrow Y \cap N = \varnothing$

$\Rightarrow N \subset X - Y.$

Thus we have shown that to each $p \in X - Y$, there exists a neighbourhood N of p such that $N \in X - Y$ and consequently $X - Y$ is open. It follows that Y is closed.

Theorem 3: Closed subsets of compact sets are compact.

Proof: Let Y be a compact subset of a metric space X and let F be a subset of Y, closed relative to X. To show that F is compact, let

$$C = \{G_\lambda : \lambda \in \Lambda\}$$

be an open cover of F. Then the collection

$$D = \{G_\lambda : \lambda \in \Lambda\} \cup \{X - F\}$$

forms an open cover of Y. Since Y is compact, there is a finite subcollection D' of D which covers Y, and hence F. If $X - F$ is a member of D', we may remove it from D' and still retain an open finite cover of F. We have thus shown that a finite subcollection of C covers F. Hence, F is compact.

Corollary 1: If F is closed and Y is compact then $F \cap Y$ is compact.

Sequentially and Countably Compact Sets

Theorem 4: A compact metric space is sequentially compact.

Proof: Let A be an infinite set in a compact metric space X. To prove that A has a limit point we must find a point p for which every open neighbourhood of p contains infinitely many points of A. Suppose that no such point exists. Then every point of X has an open neighbourhood containing only finitely many points of A. These sets form an open cover of X and extracting a finite open cover gives a covering of X meeting A in only finitely many points. This is impossible since $A \subset X$ and A is infinite.

Corollary 2: In a compact metric space every bounded sequence has a convergent subsequence.

Proof: Given the above limit point p, take x_{i1} to be in a 1-neighbourhood of p, x_{i2} to be in a 1/2 neighbourhood of p,... and we get a subsequence converging to p.

Together with the Heine-Borel theorem, this implies the Bolzano-Weierstrass theorem.

Compactness vs Sequential Compactness

K is compact if every open cover of K contains a finite subcover. K is sequentially compact if every infinite subset of K has a limit point in K.

Theorem 5: K is compact $\Leftrightarrow$ K is sequentially compact.

The proof requires the introduction of two auxiliary notions:

A space X is separable if it admits a countable dense subset.

A collection $\{V_a\}$ of open subsets of X is said to be a base for X if the following is true: for every $x \in X$ and for every open set $G \subset X$ such that $x \in G$, there exists α such that $x \in V_\alpha \subset G$.

In other words, every open subset of X decomposes as a union of a subcollection of the V_α's — the V_α's 'generate' all open subsets. The family $\{V_\alpha\}$ almost always contains infinitely many members (the only exception is if X is finite). However, if X happens to be separable, then countably many open subsets are enough to form a base (the converse statement is also true and is an easy exercise).

Lemma 1: Every separable metric space has a countable base.

Proof: Assume X is separable: by definition it contains a countable dense subset $P = \{p_1, p_2,...\}$. Consider the countable collection of neighbourhoods $\{N_r(x), r \in Q, i = 1, 2,...\}$. We will show that it is a base by checking the definition.

Consider any open set $G \subset X$ and any point $x \subset G$. Since G is open, there exists $r > 0$ such that $N_r(x) \subset G$. Decreasing r if necessary we can assume without loss of generality that r is rational. Since P is dense, by definition x is a limit point of P, so $N_{r/2}(x)$ contains a point of P. So there exists i such that $d(x, p_i) < \frac{r}{2}$.

Since r is rational, the neighbourhood $N_{r/2}(p_i)$ belongs to the chosen collection. Moreover, $N_{r/2}(p_i) \subset N_r(x) \subset G$. Finally, since $d(x, p_i) < \frac{r}{2}$ we also have $x \in N_{r/2}(p_i)$. So the chosen collection is a base for X.

Lemma 2: If X is sequentially compact then it is separable.

Proof: Fix $\delta > 0$ and let $x_1 \in X$. Choose $x_2 \in X$ such that $d(x_1, x_2) > \delta$, if possible. Having chosen $x_1,..., x_j$, choose x_{j+i} (if possible) such that $d(x_i, x_{j+i}) > \delta$ for all $i = 1,... ,j$. We first notice that this process has to stop after a finite number of iterations. Indeed, otherwise we would obtain an infinite sequence of points x_i mutually distant by at least δ; since X is sequentially compact the infinite subset $\{x_i, i = 1,2,...\}$ would admit a limit point y, and the neighbourhood $N_{\delta/2}(y)$ would contain infinitely many of the x_i's, contradicting the fact that any two of them are distant by at least d. So after a finite number of iterations we obtain $x_1,..., x_j$ such that $N_\delta(x_1)\cup... \cup N_\delta(x_j) = X$ (every point of X is at distance less than δ from one of the x_i's).

We now consider this construction for $\delta = \dfrac{1}{n}$ $(n = 1, 2,...)$. For $n = 1$ the construction gives points $x_{11},..., x_{1j1}$ such that $N_1(x_{11}) \cup......\cup N_1(x_{1j1}) = X$, for $n = 2$ we get $x_{21},..., x_{2j2}$ such that $N_{1/2}(x_{21}\cup...\cup N_{1/2}(x_{2j2})X$, and so on. Let $S = \{x_{ki},$ $k \geq 1, 1 \leq i \leq jk\}$: clearly S is countable. We claim that S is dense (i.e., $\overline{S} = X$). Indeed, if $x \in X$ and $r > 0$, the neighbourhood $N_r(x)$ always contains at least a point of S (choosing n so that $\dfrac{1}{n} < r$, one of the x_{ni}'s is at distance less than r from x), so every point of X either belongs to S or is a limit point of S, i.e., $\overline{S} = X$.

At this point we know that every sequentially compact set has a countable base. We now show that this is enough to get *countable* subcovers of any open cover.

Lemma 3: If X has a countable base, then every open cover of X admits an at most countable subcover.

Lemma 4: If $\{F_n\}$ is a sequence of non-empty closed subsets of a sequentially compact set K such that $F_n \supset F_{n+1}$ for all $n = 1, 2,...$, then $\cap_{n=1}^{\infty} F_n \neq \phi$.

Proof: Take $x_n \in F_n$ for each integer n, and let $E = \{x_n, n = 1, 2,... \}$. If E is finite then one of the x_i belongs to infinitely many F_n's. Since $F_1 \supset F_2 \supset ...$, this implies that x_i belongs to every F_n, and we get that $y \in \cap_{n=1}^{\infty} F_n$ is not empty.

Assume now that E is infinite. Since K is sequentially compact, E has a limit point y. Fix a value of n: every neighbourhood of y contains infinitely many points of E; among them, we can find one which is of the form x_i for $i > n$ and therefore belongs to F_n (because $x_i \in F_i \subset F_n$). Since every neighbourhood of y contains a point of F_n, we get that either $y \in F_n$, or y is a limit point of F_n; however, since F_n

is closed, every limit point of F_n belongs to F_n. So in either case we conclude that $y \in F_n$. Since this holds for every n, we obtain that $y \in \cap_{n=1}^{\infty} F_n$, which proves that the intersection is not empty.

We can now prove the theorem. Assume that K is sequentially compact, and let $\{G_\alpha\}$ be an open cover of K. By Lemma 1 and Lemma 2, K has a countable base, so by Lemma 3 $\{G_\alpha\}$ admits an at most countable subcover that we will denote by $\{G_i\}_{i \geq 1}$. Our aim is to show that $\{G_i\}$ admits a finite subcover (which will also be a finite subcover of $\{G_\alpha\}$. If $\{G_i\}$ only contains finitely many members, we are already done; so assume that there are infinitely many G_i's and assume that for every value of n we have $G_1 \cup ..... \cup G_n \not\subset K$ (else we have found a finite subcover).

Let $F_n = \{x \in K, x \in G_1 \cup ..... \cup G_n\} = K \cap G_n^c \cap .... \cap G_n^c$. Because the G_i are open, F_n is closed; by assumption F_n is non-empty and clearly $F_n \supset F_{n+1}$ for all n. Therefore, applying Lemma 4 we obtain that $\cap_{n=1}^{\infty} F_n \neq \phi$ and there exists a point $y \in K$ such that $y \notin G_1 \cup ..... \cup G_n$ for every n. We conclude that $y \notin \cup_{i=1}^{n} G_i$, which is a contradiction since the open sets G_i cover K.

So there exists a value of n such that $G_1,..., G_n$ cover K. We conclude that every open cover of K admits a finite subcover and therefore that K is compact.

11.2.1 The Bolzano-Weierstrass Property and Sequential Compactness

We say that x is a cluster point for a sequence (x_n) if for any $N > 0$ and any open neighbourhood U_x of x, there is an $n > N$ such that $x_n \in U_x$.

Bolzano-Weierstrass Property

A topological space X satisfies the Bolzano-Weierstrass (B-W) property if every sequence (x_n) from X has at least one cluster point.

- If X is compact, then X satisfies B-W.
- Suppose X is compact, but there is sequence (x_n) with no cluster point. Then for every $z \in X$, there is an open neighbourhood U_z of z and $N_z > 0$ such that $x_n \notin U_z$ for $n \geq N_z$. Then since $X \subset \cup_{z \in X} U_z$ and X is compact, there is a finite set $\{z_i\}_{i=1}^{m}$ such that $X \subset \{z_i\}_{i=1}^{m} U_{z_1}$. But this contradicts $x_n \notin \cup_{i=1}^{m} U_z$, for all $n \geq \max \{N_{z_i}\}_{i=1}^{m}$.

NOTES

Sequential Compactness

Suppose (x_n) is a sequence from X. Now consider any sequence (τ_n) of positive integers such that $\tau_n < \tau_{n+i}$ for all n. Then $\left(x_{\tau_n}\right) \equiv \left(x_{\tau_n}\right)_{n=1}^{\infty}$ is called a *subsequence* of (x_n), for example,

- If (x_n) is a sequence, then $(x_2, x_4, x_6, ...)$ is a subsequence of (x_n).
- If (x_n) converges, then every subsequence of (x_n) converges to the same limit.

  We say that X is *sequentially compact* if every sequence (x_n) in X has a convergent subsequence.
- X is sequentially compact if and only if it satisfies B-W.
- Consider any sequence (x_n) from X.
- Suppose X is sequentially compact. Then there is a subsequence (x_{τ_n}) such that $x_{\tau_n} \to x \in X$. Therefore, for any $\varepsilon > 0$, there is an $N > 0$ such that $n > N$ implies $d(x_{\tau_n}, x) < \varepsilon$. Therefore, $x_n \in N_\varepsilon(x)$ for all $n > \tau_N$, which implies that (x_n) satisfies B-W.
- Suppose X satisfies B-W so that there is a cluster point $x \in X$. We may construct convergent sequence recursively as follows. Let $\tau_1 = 1$. For each n choose $\tau_n \geq \tau_{n-1}$ such that $(x_{\tau_n}) \in N_{\frac{1}{n}}(x)$. Then $x_{\tau_n} \to x$.
- If X compact, then it is sequentially compact.
- Follows immediately from the fact that sequential compactness is equivalent to B-W, which is implied by compactness.

Completeness

- If (X, d) is a compact space, then it is a complete space.
- Suppose X is compact and consider any Cauchy sequence (x_n) from X. Since X is compact, it is sequentially compact and therefore there is a subsequence (x_{τ_n}) such that $x_{\tau_n} \to x$.
- We need to show that $x_n \to x$. Consider any $\varepsilon > 0$. Then (x_n) Cauchy implies an $N > 0$ such that $d(x_n, x_m) < \frac{\varepsilon}{2}$ for $n, m \geq N$. But $x_{\tau_n} \to x$ also implies that we may choose N sufficiently large so that $d(x_{\tau_n}, x) < \frac{\varepsilon}{2}$.

Then since $\tau_N \geq N$, we have, for any $n > N$, $d\{x_n, x) \leq d(x_n, x_{\tau_n}) + d(x_\tau, x) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

11.2.2 Finite Intersection Property

The finite intersection property is a property of a collection of subsets of a set *X*. A collection has this property if the intersection over any finite subcollection of the collection is nonempty.

A centered system of sets is a collection of sets with the finite intersection property.

Let *X* be a set with $A = \{A_i\}_{i \in I}$, a family of subsets of *X*. Then the collection *A* has the finite intersection property (fip), if any finite subcollection $J \subset I$ has non-empty intersection $\bigcap_{i \in J} A_i$.

Theorem 6: Let *X* be a compact Hausdorff space that satisfies the property that no one-point set is open. If *X* has more than one point, then *X* is uncountable.

Before proving this, we give some examples:

1. We cannot eliminate the Hausdorff condition; a countable set with the indiscrete topology is compact, has more than one point and satisfies the property that no one point sets are open, but is not uncountable.
2. We cannot eliminate the compactness condition as the set of all rational numbers shows.
3. We cannot eliminate the condition that one point sets cannot be open as a finite space given the discrete topology shows.

Proof: Let *X* be a compact Hausdorff space. We will show that if *U* is a nonempty, open subset of *X* and if *x* is a point of *X*, then there is a neighbourhood *V* contained in *U* whose closure does not contains *x* (*x* may or may not be in *U*). First of all, choose *y* in *U* different from *x* (if *x* is in *U*, then there must exist such a *y* for otherwise *U* would be an open one- point set; if *x* is not in *U,* this is possible since *U* is nonempty). Then by the Hausdorff condition, choose disjoint neighbourhoods *W* and *K* of *x* and *y*, respectively. Then $(K \cap U)$ will be a neighbourhood of *y* contained in *U* whose closure does not contains *x* as desired.

Now suppose *f* is a bijective function from *Z* (the positive integers) to *X*. Denote the points of the image of *Z* under *f* as $\{x_1, x_2, ...\}$. Let *X* be the first open set and choose a neighbourhood U_1 contained in *X* whose closure does not contains

x_1. Secondly, choose a neighbourhood U_2 contained in U_1 whose closure does not contains x_2. Continue this process whereby choosing a neighbourhood U_{n+1} contained in U_n whose closure does not contains x_{n+1}. Note that the collection $\{U_i\}$ for i in the positive integers satisfies the finite intersection property and hence the intersection of their closures is nonempty (by the compactness of X). Therefore there is a point x in this intersection. No x_i can belong to this intersection because x_i does not belongs to the closure of U_i. This means that x is not equal to x_i for all i and f is not surjective; a contradiction. Therefore, X is uncountable.

Corollary 3: Every closed interval $[a, b]$ $(a < b)$ is uncountable. Therefore, the set of real numbers is uncountable.

Corollary 4: Every locally compact Hausdorff space that is also perfect is uncountable.

Proof: Suppose X is a locally compact Hausdorff space that is perfect and compact. Then it immediately follows that X is uncountable (from the theorem). If X is a locally compact Hausdorff, perfect space that is not compact, then the one-point compactification of X is a compact Hausdorff space that is also perfect. It follows that the one-point compactification of X is uncountable. Therefore X is uncountable (deleting a point from an uncountable set still retains the uncountability of that set).

A collection $A\{A_\alpha\}_{\alpha\in I}$ of subsets of a set X is said to have the finite intersection property, abbreviated fip, if every finite subcollection $\{A_1, A_2,..., A_n\}$ of A satisifes $\bigcap_{i=1}^{n} A_i \neq \phi$.

The finite intersection property is most often used to give the following equivalent condition for the compactness of a topological space (a proof of which may be found here):

Proposition: A topological space X is compact if and only if for every collection $C = \{C_\alpha\}_{\alpha\in J}$ of closed subsets of X having the finite intersection property has nonempty intersection.

An important special case of the preceding condition is that in which C is a countable collection of non-empty nested sets, i.e., when we have $C_1 \supset C_2 \supset C_3 \supset$.... In this case, C automatically has the finite intersection property and if each C_i is a closed subset of a compact topological space, then, by the proposition, $\bigcap_{i=1}^{n} C_i \neq \phi$.

The fip characterization of compactness may be used to prove a general result on the uncountability of certain compact Hausdorff spaces, and is also used in a proof of Tychonoff's theorem.

Theorem 7: A topological space is compact if and only if any collection of its closed sets having the finite intersection property has non-empty intersection.

The preceding theorem is essentially the definition of a compact space rewritten using de Morgan's laws. The usual definition of a compact space is based on open sets and unions. The above characterization, on the other hand, is written using closed sets and intersections.

Proof: Suppose X is compact, i.e., any collection of open subsets that cover X has a finite collection that also covers X. Further, suppose $\{F_i\}_{i\in I}$ is an arbitrary collection of closed subsets with the finite intersection property. We claim that $\cap_{i\in I} F_i$ is non-empty.

Suppose otherwise, i.e., suppose $\cap_{i\in I} F_i = \phi$.

Then, $X = \left(\bigcap_{i\in I} F_i\right)_c \bigcap_{i\in I} (F_i)_c$ (Here, the complement of a set A in X is written as A_c). Since each F_i is closed, the collection $\{F_{i_c}\}_{i\in I}$ is an open cover for X. By compactness, there is a finite subset $J \subset I$ such that $X = \cup_{i\in J} F_{i_c}$. But then $X = \left(\cup_{i\in J} F_i\right)_c$, so $\cap_{i\in J} F_i = \phi$, which contradicts the finite intersection property of $\{F_i\}_{i\in I}$.

The proof in the other direction is analogous. Suppose X has the finite intersection property. To prove that X is compact, let $\{F_i\}_{i\in I}$ be a collection of open sets in X that cover X. We claim that this collection contains a finite subcollection of sets that also cover X. The proof is by contradiction. Suppose that $X = \cup_{i\in J} F_i$ holds for all finite $J \subset I$. Let us first show that the collection of closed subsets $\{F_{i_c}\}_{i\in I}$ has the finite intersection property. If J is a finite subset of I, then $\bigcap_{i\in J} F_{i_c} = \left(\bigcap_{i\in J} F_i\right)_c \neq \phi$ where the last assertion follows since J was finite. Then, since X has the finite intersection property, $\phi \neq \bigcap_{i\in J} F_{i_c} = \left(\bigcap_{i\in J} F_i\right)_c$, this contradicts the assumption that $\{F_i\}_{i\in I}$ is a cover for X.

NOTES

11.2.3 Continuous Functions and Compact Sets

$D \cup R$ is compact if and only if for any given open covering of D we can subtract a finite sucovering. That is, given (G_α), $\alpha \in A$, a collection of open subsets of R (A an arbitrary set of indices) such that $D \subset \cup_{\alpha\in A} G_\alpha$, then there exist finitely many indices $\alpha_1,...,\alpha_n \in A$ such that $D \subset \cup_{i=1}^{N} G_{\alpha_1}$.

Let D be an arbitrary subset of R. Then $A \subset D$ is open in D (or relative to D, or D-open) if and only if there exists G open subset of R such that $D = G \cap D$. Similarly we can define the notion of D-closed sets. Note that D is both open and closed in D and so is ϕ.

$D \subset R$ is connected if and only if ϕ and D are the only subsets of D which are both open in D and closed in D. In other words, if $D = A \cup B$ and A, B are disjoint D-open subsets of D, then either $A = \phi$ or $B = \phi$

Let $D \subseteq R$, $a \in D$ a fixed element and $f : D \to R$ an arbitrary function. By definition, f is continuous at a if and only if the following property holds:

$$\forall \varepsilon > 0, \exists \delta_a(\varepsilon) > 0$$

such that $|x-a| < \delta_a(\varepsilon)$ and $x \in D \Rightarrow |f(x) - f(a)| < \varepsilon$

The last implication can be rewritten in terms of sets as follows:

$$f\left(B_a\left(\delta_a(\varepsilon)\right) \cap D\right) \subseteq B_{f(a)}(\varepsilon)$$

Here, we use the notation $B_x(r) := (x - r, x + r)$.

Characterization of Continuous Functions Using Preimages

Theorem 8: Let $D \subseteq R$ and $f : D \to R$ a function. Then the following propositions are equivalent:

- f is continuous (on D).
- $\forall G \subseteq R$ open, $f^{-1}(G)$ is open in D.
- $\forall F \subseteq R$ closed, $f^{-1}(F)$ is closed in D.

Proof: $a \Rightarrow b$. Let $G \subset R$ open. Pick $a \in f^{-1}(G)$. Then $f(a) \in G$ and since G is open, there must exist $\varepsilon > 0$ such that $B_{f(a)}(\varepsilon) \subseteq G$. By continuity, corresponding to this $\varepsilon > 0$ there exists $\delta > 0$ such that $f\left(B_a(\delta) \cap D\right) \subset B_{f(a)}(\varepsilon)$. But this places the entire set $B_0(\delta) \cap D$ inside $f^{-1}(G)$:

$$B_a(\delta) \cap D \subseteq f^{-1}(G)$$

Writing now $\delta = \delta_a$ to mark the dependence of δ on a and varying $a \in f^{-1}(G)$, we obtain

$$f^{-1}(G) = \left(\cup_{a \in f^{-1}(G)} B_a(\delta_a)\right) \cap D$$

which shows that $f^{-1}(G)$ is open in D.

$b \Leftrightarrow c$. Let $F \subseteq R$ a closed set, which is equivalent to saying that $G = GF$ (the complement in R) is open.

Then

$$f^{-1}(F) = \{x \in D \mid f(x) \in F\} = \{x \in D \mid f(x) \notin G\} = D - f^{-1}(G)$$

Since the complement of a D-open subset of D is D-closed, it means that $f^{-1}(F)$ is closed in D if and only if $f^{-1}(G)$ is open in D.

$c \Rightarrow a$: Left as an exercise.

Using this characterization, we can prove for example that the composition of continuous functions is a continuous function.

Proposition: Assume $f : D \to R$ is continuous, $g : E \to R$ is continuous and $f(D) \subseteq E$. Then the function $h := gof : D \to M$ defined by $h(x) = g(f(x))$ is continuous.

Proof: Let $G \subseteq R$ an open set. Then $h^{-1}(G) = f^{-1}(g^{-1}(G))$. But $g^{-1}(G) = V \cap E$, for some open set $V \subseteq R$. But then $h^{-1}(G) = f^{-1}(V \cap E) - f^{-1}(V)$ is open in D. So h is continuous.

Theorem 9: Assume $f : D \to R$ is a continuous function, such that $f(x) \neq 0$, $\forall x \in D$. Then $h : D \to R$ given by $h(x) = 1/f(x)$, is continuous as well.

Proof: $g : R - \{0\} \to R$ $g(x) = 1/x$ is continuous, $f(D) \subseteq R - \{0\}$, hence $h = gof$ is continuous.

General Properties of Continuous Functions

Theorem 10: A continuous function maps compact sets into compact sets.

Proof: In other words, assume $f : D \to R$ is continuous and D is compact. Then we need to prove that the image $f(D)$ is a compact subset of R. For that, we consider an arbitrary open covering $f(D) \subseteq \cup_\alpha G_\alpha$ of $f(D)$ and we will try to find a finite subcovering. Taking the preimage we have $D \subseteq \cup_\alpha f^{-1}(G_\alpha)$. But $f^{-1}(G_\alpha)$ is open in D, so there must exist $V_a \subseteq R$ open such that $f^{-1}(G_\alpha) = V_\alpha \cap D$. Then $D \subseteq \cup_\alpha (V_\alpha \cap D)$ which simply means that $D \subseteq \cup_\alpha V_\alpha$. We thus arrived at an open covering of D. So there must exist finitely many indices $\alpha_1, ..., \alpha_N$ such that $D \subseteq$

$\cup_{i=1}^{N} G_{\alpha_i}$, which implies the equality $D = \cup_{i=1}^{N}(V_{\alpha_i} \cap D) = \cup_{i=1}^{N} f^{-1}\left(G_{\alpha_i}\right)$. But this implies in turn that $f(D) \subseteq \cup_{i=1}^{N} G_{\alpha_i}$ is compact.

Theorem 11: A continuous function maps connected sets into connected sets.

In other words, assume $f: D \to R$ is continuous and D is connected. Then $f(D)$ is connected as well.

Proof: Assume $f(D)$ is not connected. Then there must exist A, B disjoint, non-empty subsets of $f(D)$, both open relative to $f(D)$, such that $f(D) = A \cup B$. Being open relative to $f(D)$ simply means there exists $U, V \subseteq R$ open such that $A = f(D) \cap U$, $B = f(D) \cap V$. So $f(D) \subseteq U \cup V$. But this implies that $D \subset f^{-1}(U) \cap f^{-1}(V)$. Since U, V are open, it follows that $f^{-1}(U)$ and $f^{-1}(V)$ are open relative to D. But they are also disjoint. Since D is connected, it follows that at least one of them, say $f^{-1}(U)$, is empty. But $A \subseteq U$, so this forces $f^{-1}(A) = \phi$ as well, which is impossible unless $A = \phi$ (note that A is a subset of the image of f), contradiction.

Theorem 12: A continuous function on a compact set is uniformly continuous.

Proof: Assume D compact and $f: D \to R$ continuous. Given $\varepsilon > 0$ we need to find $\delta(\varepsilon) > 0$ such that if $x, y \in D$ and $|x - y| < \delta(\varepsilon)$, then $|f(x) - f(y)| < \varepsilon$.

From the definition of continuity, given $\varepsilon > 0$ and $x \in D$, there exists $\delta_x(\varepsilon)$ such that if $|y - x| < \delta_x(\varepsilon)$, then $|f(y) - f(x)| < \varepsilon$. Clearly $D \subseteq \cup_{x \in D} B_x\left(\frac{1}{2}\delta\left(\varepsilon/2\right)\right)$. From this open covering we can extract a finite subcovering (D is compact), meaning there must exists finitely many $x_1, x_2, ..., x_N$ εD such that $D \subseteq \cup_{i=1}^{N} B_{x_i}(\frac{1}{2}\delta_{x_i}(\varepsilon/2))$.

Let now $\delta(\varepsilon) = \min\{\frac{1}{2}\delta_{x_i}(\varepsilon/2)\}$.

Take $y, z \in D$ arbitrary such that $|y - z| < \delta(\varepsilon)$. The idea is that y will be near some x_j, which in turn places z near that same x_j. But that forces both $f(y)$, $f(z)$ to be close to $f(x_j)$ (by continuity at x_j), and hence close to each other.

Since $y \in D$, there must exist some j, $1 \le j \le N$ such that $y \in B_{x_j}(\frac{1}{2}\delta_{x_j}(\varepsilon/2))$. Thus

- $|y - x_j| < \frac{1}{2}\delta_{x_j}$ $(\varepsilon/2)$

- but $|y - z| < \delta(\varepsilon) \leq \frac{1}{2}\delta_{x_j}$ $(\varepsilon/2)$

By the triangle inequality it follows that $|z - x| < \delta_{x_j}$ $(\varepsilon/2)$. So y, z are within δ_{x_j} $(\varepsilon/2)$ of x.

This implies that

- $|f(y) - f(x_j)| < \varepsilon/2$
- $|f(z) - f(x_j)| < \varepsilon/2$

By the triangle inequality once again we have $|f(y) - f(z)| < \varepsilon$.

Alternative proof using sequences: Assume f is not uniformly continuous, meaning that there exists $\varepsilon > 0$ such that no $\delta > 0$ does the job.

Checking what this means for $\delta = \frac{1}{n}$, we see that for any such $3n \geq 1$ there exist $x_n, y_n \in D$ such that $|x_n - y_n| < \frac{1}{n}$ and yet $|f(x_n) - f(y_n)| > \varepsilon$. However D is compact; in particular any sequence in D has a convergent subsequence whose limit belongs to D. Applying this principle twice we find that there must exist $n_1 < n_2 < ...$ such that the subsequences $\left(x_{n_k}\right)_{k\geq 1}$ and $\left(y_{n_k}\right)_{k\geq 1}$ are convergent, and $x = \lim_{x\to\infty} x_{n_k} \in D$, $y = \lim_{x\to\infty} y_{n_k} \in D$. We have the following:

- By construction, $\left|x_{n_k} - y_{n_k}\right| < \frac{1}{n_k} \leq \frac{1}{k}$. Taking the limit, we find $x = y$.

- By continuity, $\lim_{k\to\infty} f\left(x_{n_k}\right) = f(x)$, since $x \in D$. Also

  $\lim_{k\to\infty} f\left(y_{n_k}\right) = f(y)$.

- Also by construction, $\left|f\left(x_{n_k}\right) - \left(y_{n_k}\right)\right| > \varepsilon$. Hence in the limit, $|f(x) - f(y)| > \varepsilon$.

We thus reach a contradiction.

NOTES

Proposition: Let $D \subseteq R$. Then the following propositions are equivalent:

a) D is compact
b) D is bounded and closed
c) Every sequence in D has a convergent subsequence whose limit belongs to D.

Proof: $a \Rightarrow b$. $D \subseteq R = \cup_{n=1}\infty(-n, n)$ is an open covering of that D. Hence $\exists N \geq 1$ such that $D \subseteq \cup_{n=1}\infty(-n, n)$ = $(-N, N)$. This shows that D is bounded. To prove D is closed, we prove that $R - D$ is open. Let $y \in R - D$. Then $D \subseteq \cup_{n=1}^{\infty}\left(R - [y - \frac{1}{n}, \frac{1}{n}]\right)$. This open covering must have a finite subcovering, so $\exists N \geq 1$ such that $D \subseteq R - [y - \frac{1}{N}, y + \frac{1}{N}]$. But this implies that $\left(y - \frac{1}{N}, y + \frac{1}{N}\right) \subseteq R - D$. But y was chosen arbitrary in $R - D$, so this set is open, and hence D itself is closed.

$b \Rightarrow c$. This has to do with the fact that every bounded sequence has a convergent subsequence.

$c \Rightarrow b$. Here one shows $D = \bar{D}$ and this has to do with the fact that D is the set of limits of convergent sequences of D, etc.

c $\Rightarrow a$. Let $D \subseteq \cup_{k=1}^{\infty} G_k$ be an arbitrary open covering of D.

Note: A covering by a countable collection of open sets is not the most general infinite open covering one can imagine, of course; we need an intermediate step to prove that from any open covering of D we can extract a countable subcovering, and this has to do with the fact that R admits a countable dense set.

We will now prove that there exists $n \geq 1$ such that $D \subseteq \cup_{k=1}^{\infty} G_k$. Assume this was not the case. Then $\forall N \geq 1$, there exists $x_n \in D - \cup_{k=1}^{\infty} G_k$. But x_n is a sequence in D, so it must have a convergent subsequence; call it $\left(x_{n_j}\right)_{j \geq 1}$, with limit in D. So $\lim_{j \to \infty} x_{n_j} = a \in D$. But a belongs to one of the G_i's, say $a \in G_N$. Since G_N is open, it follows that $x_{n_j} \in G_N$, for $j \geq j_0$ (j large enough). In particular this shows that for j large enough (larger than j_0 and larger than N) we have $x_{n_j} \in G_N \subseteq \cup_{k=1}^{n_j}$,

since $n_j \geq j > N$. This contradicts the defining property of x_n's.

NOTES

Theorem 13: R is connected.

Proof: This can be restated as, ϕ and R itself are the only subsets of R which are both open and closed. To prove this, let E be a non-empty subset of R with this property. We will prove that $E = R$. For that, take an arbitrary $c \in R$. To prove that $c \in E$, we assume that $c \notin E$ and look for a contradiction. Since E is non-empty, it follows that E either has points to the left of c or to the right of c. Assume that the former holds.

- Consider the set $S = \{x \in E | x < c\}$. By construction, S is bounded from above (c is an upper bound for S). Therefore we can consider $y = lub(S) \in R$.
- Input: E is closed. Then $S = E \cap (-\infty, c]$ is also closed. Then $y \in \overline{S} = S$, so $y < c$.
- Input: E is open, $y \in S \subseteq E$ and E is open. This means that there exists $\varepsilon > 0$ such that $(y - \varepsilon, y + \varepsilon) \subseteq E$. Choose ε small enough so that $\varepsilon < c - y$. In that case $z = y + \varepsilon/2 \in (y - \varepsilon, y + \varepsilon) \subseteq E$ is an element of E with the properties,
  - $z < c$, hence $z \in S$
  - $z > y$

which is in contradiction with the defining property of y.

Theorem 14: The only connected subsets of R are the intervals (bounded or unbounded, open or closed or neither).

Proof: First we prove that a connected subset of R must be an interval.

Let $E \subseteq R$ be a connected subset. We prove that if $a < b \in E$, then $[a,b] \subseteq E$. In other words, together with any two elements, E contains the entire interval between them. To see this, let c be a real number between a and b. Assume $c \notin E$. Then $E = A \cup B$, where $A = (-\infty, c) \cap E$ and $B = (c, +\infty) \cap E$. Note that A and B are disjoint subsets of D, both open relative to D. Since S is connected, at least one of them should be empty, contradiction, since $a \in A$ and $b \in B$. Thus $c \in E$.

To show that E is actually an interval, consider inf E and sup E. E is bounded. Then $m = \inf E$, $M = \sup E \in R$, and clearly $E \subseteq [m, M]$. On the other hand, for any given $x \in (m, M)$, there exists $a,b \in E$ such that $a < x < b$. That is because $m, M \in E$ and one can find elements of E as close to m (respectively M) as desired (draw a picture with the interval (m, M) and place a point x inside it). But then $[a,b] \subseteq E$, and in particular $x \in E$. Since x was chosen arbitrarily in (m, M), we must have $(m, M) \subseteq E \subseteq [m, M]$, so E is definitely an interval. Case two: E is unbounded. With a similar argument, show that E is an unbounded interval.

NOTES

Conversely, we need to show that intervals are indeed connected sets. The proof is almost identical to that in the case where the interval is R itself.

Theorem 15: Let $D \subseteq R$ be compact and $f: D \to R$ be a continuous function. Then there exists $y_1, y_2 \in D$ such that $f(y_1) \leq f(x) \leq f(y_2)$, $\forall x \in D$.

Proof: $f(D)$ is a compact subset of R, so it is bounded and closed. This implies that glb$(f(D)) \in f(D)$ and lub$(f(D)) \in f(D)$ as well. But then there must exist $y_1, y_2 \in D$ such that $f(y_1) = \text{glb} f(D)$ and $f(y_2) = \text{lub} f(D)$. But this implies $f(D) \subseteq [f(y_1), f(y_2)]$ and we are done.

Note: We use the notation $\sup_{x \in D} f(x)$ to denote the lub of the image of D. In other words, $\sup_{x \in D} f(x) = \text{lub}\{f(y) \mid y \in D\}$. The theorem says that if D is compact and f is continuous, then $\sup_{x \in D} f(x)$ is finite, and moreover that there exists $y_1 \in D$ such that $f(y_1) = \sup_{x \in D} f(y)$. If the domain is not compact, one can find examples of continuous functions such that either i) sup $f = +\infty$ or such that ii) sup f is a real number but not in the image of f.

For case i), take $f(x) = 1/x$ defined on $(0,1]$. For case ii), take $f(x) = x$ defined on $[0,1)$.

Theorem 16: A continuous (real-valued) function defined on an interval in R has the intermediate value property.

Proof: Assume E is an interval in R and $f: E \to R$ a continuous function. Let $a,b \in E$ (say $a < b$) and y a number between $f(a)$ and $f(b)$. The intermediate value property is the statement that there exists c between a and b such that $f(c) = y$. But this follows immediately from the fact that $f(E)$ is an interval. E is an interval in R $\Rightarrow E$ is connected $\Rightarrow f(E)$ is a connected subset of $R \Rightarrow f(E)$ is an interval in R.

Check Your Progress

1. When is a subset of a metric space compact?
2. What is the base for a space X?
3. What is a cluster point for a sequence?
4. What is the subsequence for a sequence from a topological space?

11.3 LIMIT POINT COMPACTNESS

Theorem 17: Let f be a continuous mapping of a compact metric space X into a metric spce Y. Then $f[X]$ is compact.

In other words, continuous image of a compact space is compact.

Proof: Let $\{H_\lambda : \lambda \in A\}$ be an open cover of $f[X]$. Since f is continuous, $f^{-1}[H_\lambda]$ is an open set in X. It follows that the collection $\{f^{-1}[H_\lambda] : \lambda \in A\}$ is an open cover of X. Since X is compact, there exist finitely many indices $\lambda_1, \ldots, \lambda_n$ such that

$$X = f^{-1}[H_{\lambda 1}] \cup \ldots \cup f^{-1}[H\lambda_n) = f^1[H_{\lambda 1} \ldots H\lambda_n].$$

It follows that

$$f = [X] = f[f^{-1}(H_{\lambda 1} \cup \ldots \cup H_{\lambda 1}] \subset H_{\lambda n}) \cup \ldots \cup H_{\lambda n}$$

Hence $f[X]$ is compact.

A mapping f of a set A into R^n is said to be bounded if there exists a real number M such that $|f(x)| \leq M$ for all $x \in A$.

Limits and Continuity

Let (X, d) and (Y, p) be two metric spaces and $A \subset X$. Suppose f maps A into Y, and p is a limit point of A. Then we say that $f(x)$ tends to the limit $q \in Y$ as x tends to p if for every $q > 0$ there exists a $\delta > 0$ such that

$$0 < d(x, p) < \delta \Rightarrow \rho(f(x), q) < \in$$

We then write $f(x) \to q$ as $x \to p$ or $\lim_{x \to p} f(x) = q$.

Let (X, d) and (Y, p) be two metric spaces, $A \subset X$, $p \in A$ and f maps A into Y. Then we say that f is **continuous** at p if for every $\in > 0$ there exists a $a\delta > 0$ such that

$$x \in A,\ d(x, p) < \delta \Rightarrow \rho(f(x), f(p)) < \in$$

If f is continuous at every point of A, then f is said to be **continuous** on A or simply **continuous.**

Continuity in $\mathbf{R}^n$

Let A be a subset of R^n and let **f** be a mapping of A into $\mathbf{R}^m$**.** Then **f** is said to be continuous at a point $p \in A$ if for every $\in > 0$ there exists $\delta > 0$ such that

$$|\mathbf{x} - \mathbf{p}| < \delta \Rightarrow |\mathbf{f(x)} - \mathbf{f(p)}| < \in.$$

Example 1: Let (X, d) and (Y, p) be two metric spaces and p an isolated point of $A \subset X$. Show that the map f: A $\to$ Y is continuous at p.

Solution: Let $\in > 0$ be given. Since p is an isolated point of A, we can always find a $\delta > 0$ such that the only point $x \in A$ for which $d(x, p) < \delta$ is $x = p$. We then have

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$\rho(f(x), f(p) = \rho(f(p), f(p)) = 0 < \in$.

Hence f is continuous at p.

Theorem 18: Let (X, d) (Y, ρ) and (Z, σ) be three metric spaces, $A \subset X$ f maps A into Y, g maps f[A] into Z, and h is a mapping of A into Z defined by

$h(x) = g(f(x))(x \in A)$

(The mapping h is called the **composition map** of the functions f and g). If is continuous at a point $p \in A$ and if g is continuous at $f(p)$ then h is continuous at p.

Proof: Let $\in > 0$ be given. Since g is continuous at $f(p)$ there exists $\eta > 0$ such that

$y \in f[A], \rho(y, f(p)) < \eta \Rightarrow \sigma(g(y), g(f(p)) < \in$

Again since f is continuous at p there exists $\delta > 0$ such that

$x \in A,\ d(x, p) < \delta \Rightarrow \rho(f(x), f(p)) < \eta$

Since $f(x) \in f[A]$ taking $y = f(x)$ in the equation, it then follows from the above equations that

$x \in A,\ \ d(x, p) < \delta \Rightarrow \sigma(g(f(x)), g(f(p))) < \in$

$\Rightarrow \sigma(h(x), h(p)) < \in$ by def. of h.

This shows that h is continuous at p.

Theorem 19: Let(X, d) and (Y, ρ) be two metric spaces. A mapping f of X into Y is continuous X on if and only if f^{-1} [H] is open in X for every open set H in Y.

Proof: Let f be continuous on X and H an open set in Y. To show that f^{-1} [H] is open in X. If f^{-1} [H] = ϕ, then f^{-1} [H] is surely open. If f^{-1} [H] $\neq \phi$, let $p \in f^{-1}[H]$ be arbitrary. Then $f(p) \in H$. Since H is open, there exists $\in > 0$ such that

$N(f(p), \in) \subset H$

that is, $\rho(y, f(p)) < \in \Rightarrow y \in H$.

Again since f is continuous at p, there exists $\delta > 0$ such that

$d(x, p) < \delta \Rightarrow \rho(f(x), f(p)) < \in$.

From (1) and (2), we conclude that

$d(x, p) < \delta \Rightarrow \rho(f(x) \in H \Rightarrow x \in f^{-1}[H]$

that is $N(p, \delta) \subset f^{-1}[H]$.

Thus we have shown that f^{-1} [H] contains an *nhd* of each of its points and consequently it is open.

Conversely, let f^{-1} [H] be open in X for every open set H in Y and let p be an arbitrary point of X. We have to show that f is continuous at p. Let $\in > 0$ be given. Let

$$V = \{y \in Y : \rho(y, f(p) < \in\} .$$

Then V is an open set in Y and so by hypothesis f^{-1} [V] is an open set in X and it contains p. Hence, by definition of open set, there exists, $\delta > 0$ such that

$$N(p,\delta) \subset f^{-1}[V] \qquad = d\left(\tfrac{1}{2}\ (x_1, x_2), (y_1, y_2)\right)$$

$$= \left(\left(\tfrac{1}{2} x_1, x_2\right), (y_1, y_2), \left(\tfrac{1}{2} y_1, \tfrac{1}{2}\ y_2\right)\right)$$

$$= \sqrt{\ } \left[\left(\tfrac{1}{4}\ (x_1 - x_2)^2 + \tfrac{1}{4}\ (y_1 - y_2)^2\right]\right.$$

$$= \tfrac{1}{2} \sqrt{[(x_1 - x_2)^2 + (y_1 - y_2)]^2}$$

$$= \tfrac{1}{2} d(x_1 - y)$$

11.4 LOCAL COMPACTNESS

Many of important spaces occuring in analysis are not compact, but instead have a local version of compactness. For example, Euclidean n-space is not compact, but each point of E^n has a neighbourhood whose closure is compact.

Definition: A topological space (X, T) is locally compact if each point has a relatively compact neighbourhood.

For example, E^n is locally compact; $\overline{B_{\rho n}(x,1)}$ is compact for each $x \in E^n$. Note that this example also shows that a locally compact subset of a Hausdorff space need not be closed. Also, any infinite discrete space is locally compact, but not compact. The set of rationals in E^1 is not a locally compact space.

Theorem 20: If (X, T) is a compact topological space, then X is locally compact.

Theorem 21: The following four properties are equivalent:

1. X is a locally compact Hausdorff space.
2. For each $x \in X$ and each neighbourhood $U(x)$, there is a relatively compact open V with $x \in V \subset \overline{V} \subset U$.

3. For each compact C and open $U \supset C$, there is a relatively compact open V with $C \subset V \subset \overline{V} \subset U$.
4. X has a basis consisting of relatively compact open sets.

NOTES

Proof

(1) $\Rightarrow$ (2) There is some open W with $x \in W \subset \overline{W}$ and $\overline{W}$ compact. Since $\overline{W}$ is a regular space and $\overline{W} \cap U$ is a neighbourhood of x in $\overline{W}$, there is a set G open in $\overline{W}$ such that $x \in G \subset \overline{G_{\overline{W}}} \subset \overline{W} \cap U$. Now $G = E \cap \overline{W}$, where E is open in X and the desired neighbourhood of x in X is $V = E \cap W$.

(2) $\Rightarrow$ (3) For each $c \in C$ find a relatively compact neighbourhood V(c) with $\overline{V(c)} \subset U$; since U is compact, finitely many of these neighbourhoods cover C and therefore this union has compact closure.

(3) $\Rightarrow$ (4) Let $\mathcal{B}$ be the family of all relatively compact open sets in X; since each $x \in X$ is compact, (3) asserts that $\mathcal{B}$ is a basis.

(4) $\Rightarrow$ (1) is trivial.

Hence proved.

Local compactness is not preserved by continuous surjections. For example, if (X, T) is any non-locally compact space, map $f: X \to X$, the identity map. Then f is a (D, T) continuous surjection, where D is the discrete topology on X. But while (X, D) is locally compact, (X, T) is not by choice.

However, a continuous open map onto a Hausdroff space does preserve local compactness.

Theorem 22: Let (Y, U) be a Hausdroff space. Let $f : X \to Y$ be a (T, U) continuous, open surjection. Let X be T-locally compact, then Y is U-locally compact.

Proof: For given $y \in Y$ choose $x \in X$ so that $f(x) = y$ and choose a relatively compact neighbourhood $U(x)$. Because f is an open map, $f(U)$ is a neighbourhood of y, and because $f(\overline{U})$ is compact, we find from $\overline{f(U)} \subset \overline{f(\overline{U})} = f(\overline{U})$ that $\overline{f(U)}$ is compact.

Definition: Let (X, T) be a topological space and $A \subset X$. Then A is T-locally compact if and only if A is T_A locally compact.

Theorem 23: Let (X, T) be a local compact space and $A \subset X$. Then A is locally compact if and only if for any $x \in A$ there exists a T-neighbourhood V of x such

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that $A \cap (\overline{A \cap V})$ is T-compact.

Example 2: A subspace of a locally compact space is not necessarily compact. For example, the set if irrationals $\mathcal{I}$ is not locally compact in E^1, although E^1 is locally compact.

To see this, let V be any neighbourhood of π. If $\mathcal{I} \cap (\overline{\mathcal{I} \cap V})$ is compact then $\mathcal{I} \cap (\overline{\mathcal{I} \cap V})$ is bounded and closed. For some $\varepsilon > 0$, $]\pi-\varepsilon, \pi+\varepsilon[\subset V$. Choose any rational y with $\pi-\varepsilon < y < \pi+\varepsilon$. Then y is a cluster point of $\mathcal{I} \cap (\overline{\mathcal{I} \cap V})$, but $y \notin \mathcal{I} \cap (\overline{\mathcal{I} \cap V})$ as $y \notin \mathcal{I}$. This means that $\mathcal{I} \cap (\overline{\mathcal{I} \cap V})$ is not closed after all, a contradiction. Then $\mathcal{I} \cap (\overline{\mathcal{I} \cap V})$ cannot be compact, so $\mathcal{I}$ is not locally compact.

As with subspaces, a product of locally compact spaces need not be locally compact. If, however, the coordinate spaces are Hausdorff, and if enough of them are compact, then the product will be locally compact.

Theorem 24: $\prod \{Y_\alpha : \alpha \in \mathcal{A}\}$ is locally compact if and only if all the Y_α are locally compact Hausdroff spaces and at most finitely many are not compact.

Proof: Assume the condition holds. Given $\{y_\alpha\} \in \prod_\alpha Y_\alpha$, for each of the atmost finitely many indices for which Y_α is not compact, choose a relatively compact neighbourhood $V_{\alpha_i}(y_{\alpha_i})$; then $< V_{\alpha_1},..., V_{\alpha_n} >$ is a neighbourhood of $\{y_\alpha\}$ and $\underline{< V_{\alpha_1},..., V_{\alpha_n} >} = < \overline{V_{\alpha_1}},..., \overline{V_{\alpha_n}} >$ is compact.

Conversely, assume $\prod_\alpha Y_\alpha$ to be locally compact; since each projection p_α is a continuous open surjection, each Y_α is certainly locally compact. But also, choosing any relatively compact open $V \subset \prod_\alpha Y_\alpha$, each $p_\alpha(\overline{V})$ is compact and since $p_\alpha(\overline{V}) = Y_\alpha$ for all but atmost finitely many indices α, the result follows.

A one point compactification of a non-compact space is Hausdorff space exactly when the space is Hausdorff and locally compact.

Theorem 25: Let (X, T) be a non-compact space and (Y, U) be an Alexandroff one-point compactification of (X, T). Then (Y, U) is a Hausdorff space if and only if (X, T) is Hausdorff and locally compact.

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Theorem 26: Every locally compact Hausdorff space is completely regular.

Proof: Let (X, T) be a locally compact Hausdorff space and the space (Y, U) be an Alexandroff one-point compactification of (X, T). Since (Y, U) is a compact Hausdorff space, then U is a normal topology. By Urysohn's lemma, (Y, U) must be completely regular. Then (X, T) is a subspace of a completely regular space and so is also completely regular.

Check Your Progress

5. Define a locally compact space.
6. When does a one point compactification of a non-compact space is Hausdorff space?

11.5 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1. A subset of a metric space X is said to be compact if every open cover of A has a finite subcover, that is, if for every collection $\{G_\lambda : \lambda \in A\}$ of open sets for which

   $A \subset \cup \{G_\lambda : \lambda \in A\}$,

   there exist finitely many sets $G_{\lambda 1},...,G_{\lambda n}$ among the G_λ's such that $A \subset G_{\lambda_1} \cup ... \cup G_{\lambda_n}$.
2. A collection $\{V_a\}$ of open subsets of X is said to be a base for X if the following is true: for every $x \in X$ and for every open set $G \subset X$ such that $x \in G$, there exists α such that $x \in V_\alpha \subset G$.
3. x is a cluster point for a sequence (x_n) if for any $N > 0$ and any open neighbourhood U_x of x, there is an $n > N$ such that $x_n \in U_x$.
4. Suppose (x_n) is a sequence from X. Now consider any sequence (τ_n) of positive integers such that $\tau_n < \tau_{n+i}$ for all n. Then $\left(x_{\tau_n}\right) \equiv \left(x_{\tau_n}\right)_{n=1}^{\infty}$ is called a subsequence of (x_n)
5. A topological space (X, T) is locally compact if each point has a relatively compact neighbourhood.
6. A one point compactification of a non-compact space is Hausdorff space exactly when the space is Hausdorff and locally compact.

11.6 SUMMARY

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- By an open cover of a subset A of a metric space X, we mean a collection $C = \{G_\lambda : \lambda \in 1\}$ of open subsets of X such that $A \subset \cup \{G_\lambda : \lambda \in A\}$. We then say that C covers A.
- By a subcover of an open cover C of A, we mean a subcollection C' *of* C' such that C' covers A.
- In particular, the metric space X is said to be compact if for every collection $\{G_\lambda : \lambda \in A;$ of open sets for which

  $X = \cup\{G_\lambda : \lambda \in A\}$,

  there exist finitely many sets $G_{\lambda_1}, ..., G_{\lambda_n}$ among the G_λ's such that $X = G_{\lambda_1} \cup ... \cup G_{\lambda_n}$
- Every compact subset of a metric space is closed.
- A compact metric space is sequentially compact.
- In a compact metric space every bounded sequence has a convergent subsequence.
- K is compact if every open cover of K contains a finite subcover. K is sequentially compact if every infinite subset of K has a limit point in K.
- A collection $\{V_a\}$ of open subsets of X is said to be a base for X if the following is true: for every $x \in X$ and for every open set $G \subset X$ such that $x \in G$, there exists α such that $x \in V_\alpha \subset G$.
- Every open subset of X decomposes as a union of a subcollection of the V_α's — the V_α's 'generate' all open subsets. The family $\{V_\alpha\}$ almost always contains infinitely many members (the only exception is if X is finite). However, if X happens to be separable, then countably many open subsets are enough to form a base (the converse statement is also true and is an easy exercise).
- Every separable metric space has a countable base.
- If X is sequentially compact then it is separable.
- If X has a countable base, then every open cover of X admits an at most countable subcover.
- If $\{F_n\}$ is a sequence of non-empty closed subsets of a sequentially compact set K such that $F_n \supset F_{n+1}$ for all $n = 1, 2, ...,$ then $\cap_{n=1}^{\infty} F_n \neq \phi$.

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- A topological space X satisfies the Bolzano-Weierstrass (B-W) property if every sequence (x_n) from X has at least one cluster point.
- Suppose (x_n) is a sequence from X. Now consider any sequence (τ_n) of positive integers such that $\tau_n < \tau_{n+i}$ for all n. Then $\left(x_{\tau_n}\right) \equiv \left(x_{\tau_n}\right)_{n=1}^{\infty}$ is called a subsequence of (x_n)
- A centered system of sets is a collection of sets with the finite intersection property.
- Let X be a set with $A = \{A_i\}_{i \in I}$, a family of subsets of X. Then the collection A has the finite intersection property (fip), if any finite subcollection $J \subset I$ has non-empty intersection $\bigcap_{i \in J} A_i$.
- Let X be a compact Hausdorff space that satisfies the property that no one-point set is open.
- Every closed interval $[a, b]$ $(a < b)$ is uncountable. Therefore, the set of real numbers is uncountable.
- Every locally compact Hausdorff space that is also perfect is uncountable.
- A topological space X is compact if and only if for every collection $C = \{C_\alpha\}_{\alpha \in J}$ of closed subsets of X having the finite intersection property has nonempty intersection.
- $D \cup R$ is compact if and only if for any given open covering of D we can subtract a finite sucovering.
- Let $D \subseteq R$ and $f : D \to R$ a function. Then the following propositions are equivalent:

  o f is continuous (on D).

  o $\forall G \subseteq R$ open, $f^{-1}(G)$ is open in D.

  o $\forall F \subseteq R$ closed, $f^{-1}(F)$ is closed in D.
- Assume $f : D \to R$ is continuous, $g : E \to R$ is continuous and $f(D) \subseteq E$. Then the function $h := gof : D \to M$ defined by $h(x) = g(f(x))$ is continuous.
- A continuous function on a compact set is uniformly continuous.
- Let $D \subseteq R$. Then the following propositions are equivalent:

  (a) D is compact

  (b) D is bounded and closed

(c) Every sequence in D has a convergent subsequence whose limit belongs to D.

- The only connected subsets of R are the intervals (bounded or unbounded, open or closed or neither).
- Let $D \subseteq R$ be compact and $f : D \to R$ be a continuous function. Then there exists $y_1, y_2 \in D$ such that $f(y_1) \leq f(x) \leq f(y_2)$, $\forall x \in D$.
- A continuous (real-valued) function defined on an interval in R has the intermediate value property.
- Let f be a continuous mapping of a compact metric space X into a metric spce Y. Then $f[X]$ is compact.

  In other words, continuous image of a compact space is compact.
- If f is continuous at every point of A, then f is said to be continuous on A or simply continuous.
- Let A be a subset of R^n and let **f** be a mapping of A into $\mathbf{R}^m$. Then **f** is said to be continuous at a point $p \in A$ if for every $\in > 0$ there exists $\delta > 0$ such that

  $|\mathbf{x} - \mathbf{p}| < \delta \Rightarrow |\mathbf{f(x)} - \mathbf{f(p)}| < \in.$
- Let (X, d) (Y, ρ) and (Z, σ) be three metric spaces, $A \subset X$ f maps A into Y, g maps f[A] into Z, and h is a mapping of A into Z defined by

  $h(x) = g(f(x))(x \in A)$
- Let(X, d) and (Y, ρ) be two metric spaces. A mapping f of X into Y is continuous X on if and only if f^{-1} $[H]$ is open in X for every open set H in Y.
- If (X, T) is a compact topological space, then X is locally compact.
- The following four properties are equivalent:

  1. X is a locally compact Hausdorff space.

  2. For each $x \in X$ and each neighbourhood $U(x)$, there is a relatively compact open V with $x \in V \subset \overline{V} \subset U$.

  3. For each compact C and open $U \supset C$, there is a relatively compact open V with $C \subset V \subset \overline{V} \subset U$.

  4. X has a basis consisting of relatively compact open sets.
- Let (Y, U) be a Hausdroff space. Let $f : X \to Y$ be a (T, U) continuous, open surjection. Let X be T-locally compact, then Y is U-locally compact.
- Let (X, T) be a topological space and $A \subset X$. Then A is T-locally compact if and only if A is T_A locally compact.

- Let (X, T) be a local compact space and $A \subset X$. Then A is locally compact if and only if for any $x \in A$ there exists a T-neighbourhood V of x such that $A \cap (\overline{A \cap V})$ is T-compact.
- $\prod \{Y_\alpha : \alpha \in \mathcal{A}\}$ is locally compact if and only if all the Y_α are locally compact Hausdroff spaces and at most finitely many are not compact.
- Let (X, T) be a non-compact space and (Y, U) be an Alexandroff one-point compactification of (X, T). Then (Y, U) is a Hausdorff space if and only if (X, T) is Hausdorff and locally compact.

11.7 KEY WORDS

- **Limit point compactness:** A topological space X is said to be limit point compact if every infinite subset of X has a limit point in X.
- **Local compactness:** A topological space is locally compact if each point has a relatively compact neighbourhood.
- **Finite intersection property:** It is a property of a collection of subsets of a set X. A collection has this property if the intersection over any finite sublocation of the collection is nonempty.

11.8 SELF ASSESSMENT QUESTIONS AND EXERCISES

Short Answer Questions

1. Describe compact set in the real line.
2. Give an analytical description of compactness and sequential compactness.
3. Explain the finite intersection property.
4. Discuss general properties of continuous functions.

Long Answer Questions

1. Discuss Bolzano-Weierstrass property and sequential compactness.
2. Give a detailed account of limit point compactness.
3. Give a detailed account of continuous functions and compact sets.
4. Explain limit point compactness in detail with the help of examples.
5. Discuss local compactness in detail with the help of examples.

11.9 FURTHER READINGS

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BLOCK - IV

COUNTABILITY AXIOMS AND NORMAL SPACES

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UNIT 12 COUNTABILITY AND SEPARATION AXIOMS

Structure

12.0 INTRODUCTION

An axiom of countability is a property of distinct mathematical objects that asserts the existence of a countable set with certain properties. Without such an axiom, such a set might not provably exist. In topology and related fields of mathematics, there are several restrictions that one often makes on the kinds of topological spaces that one wishes to consider. Some of these restrictions are given by the separation axioms. These are sometimes called Tychonoff separation axioms, after Andrey Tychonoff. In this unit, you will study the axioms of countability and the separation axioms. You will understand Hausdorff spaces, principle of extension of identities and convergence of T_0 space. You will discuss first and second countable spaces. You will understand Lindelof's theorems and characterization of compact sets on R, separable spaces, separability.

12.1 OBJECTIVES

After going through this unit, you will be able to:

- Discuss the axioms of countability and the separation axioms

- Describe Hausdorff spaces, principle of extension of identities and convergence of T_0 space
- Explain first and second countable spaces
- Interpret Lindelof's theorems and characterization of compact sets on R
- Discuss separable spaces and separability

12.2 AXIOMS OF COUNTABILITY

Following are the axioms of countability specifically used for topological spaces.

First Axiom of Countability

A **topological space** X is said to satisfy the **First Axiom of Countability** if, for every $x \in X$ there exists a countable collection $\mathcal{U}$ of neighbourhoods of x, such that if N is any neighbourhood of x, then there exists $U \in \mathcal{U}$ with $U \subseteq N$.

A **topological space** that satisfies the **first axiom of countability** is said to be **First Countable**.

All **metric spaces** satisfy the first axiom of countability because for any neighbourhood N of a point x, there is an open ball $B_r(x)$ within N, and the countable collection of neighbourhoods of x that are $B_{1/k}(x)$ where $k \in \mathbb{N}$, has the neighbourhood $B_{1/n}(x)$ where $\frac{1}{n} < r$.

Theorem 1. If a topological space satisfies the first axiom of countability, then for any point x of closure of a set S, there is a sequence $\{a_i\}$ of points within S which converges to x.

Proof: Let $\{A_i\}$ be a countable collection of neighbourhoods of x such that for any neighbourhood N of x, there is an A_i such that $A_i \subset N$. Define,

$$B_n = \bigcap_{i=1}^{n} A_n .$$

Then form a sequence $\{a_i\}$ such that $A_i \subset B_i$. Consequently, $\{a_i\}$ converges to x.

Theorem 2. Let X be a topological space satisfying the first axiom of countability. Then, a subset A of X is closed if and only if all convergent sequences $\{x_n\} \subset A$ which converge to an element of X converge to an element of A.

Proof: Suppose that $\{x_n\}$ converges to x within X. The point x is a limit point of $\{x_n\}$ and thus is a limit point of A, and since A is closed, it is contained within A. Conversely, suppose that all convergent sequences within A converge to an element within A, and let x be any point of contact for A. Then by the theorem above, there is a sequence $\{x_n\}$ which converges to x, and so x is within A. Thus, A is closed.

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Second Axiom of Countability

A **topological space** is said to satisfy the **Second Axiom of Countability** if it has a **countable base**. Consequently, the topological space that satisfies the second axiom of countability is said to be **Second Countable**.

Fundamentally, the topological space that satisfies the second axiom of countable is first countable, since the countable collection of neighbourhoods of a point can be all neighbourhoods of the point within the countable base, so that any neighbourhood N of that point must contain at least one neighbourhood A within the collection, and A must be a subset of N.

Theorem 3. If a topological space X satisfies the second axiom of countability, then all open covers of X have a countable subcover.

Proof: Let $\mathcal{G}$ be an open cover of X, and let B be a countable base for X. B covers X. For all points x, select an element of $\mathcal{G}$, C_x which contains x, and an element of the base B_x, which contains x and is a subset of C_x (which is possible because B is a base). $\{B_x\}$ forms a countable open cover for X. For each B_x, select an element of $\mathcal{G}$ which contains B_x, and this is a countable subcover of $\mathcal{G}$.

Check Your Progress

1. When does a topological space satisfy the first axiom of countability?
2. What is meant by a first countable topological space?

12.3 THE SEPARATION AXIOMS

The separation axioms T_i specify the degree to which distinct points or closed sets may be separated by open sets. These axioms are statements about the richness of topology.

Definition (T_i axioms)**:** Let $(X, \mathcal{T})$ be a topological space.

T_0 axiom : If a, b are two distinct elements in X, there exists an open set $U \in \mathcal{T}$ such that either $a \in U$ and $b \notin U$, or $b \in U$ and $a \notin U$ (i.e., U containing exactly one of these points).

T_1 axiom: If $a, b \in X$ and $a \neq b$, there exist open sets $U_a, U_b \in \mathcal{T}$ containing a, b respectively, such that $b \notin U_a$, and $a \notin U_b$.

T_2 axiom: If $a, b \in X$, $a \neq b$, there exist disjoint open sets $U_a, U_b \in \mathcal{T}$ containing a, b respectively.

T_3 axiom: If A is a closed set and b is a point in X such that $b \notin A$, there exist disjoint open sets $U_A, U_b \in \mathcal{T}$ containing A and b respectively.

T_4 axiom : If A and B are disjoint closed sets in X, there exist disjoint open sets $U_A, U_B \in \mathcal{T}$ containing A and B respectively.

T_5 axiom : If A and B are separated sets in X, there exist disjoint open sets U_A , $U_B \in \mathcal{T}$ containing A and B respectively.

If $(X, \mathcal{T})$ satisfies a T_i axiom, X is called a T_i space. A T_0 space is sometimes called a Kolmogorov space and a T_1 space, a Frechet space. A T_2 is called a Hausdorff space.

Each of axioms in Definition above is independent of the axioms for a topological space; in fact there exist examples of topological spaces which fail to satisfy any T_i. But they are not independent of each other, for instance, axiom T_2 implies axiom T_1, and axiom T_1 implies T_0.

More importantly than the separation axioms themselves is the fact that they can be employed to define successively stronger properties. To this end, we note that if a space is both T_3 and T_0 it is T_2, while a space that is both T_4 and T_1 must be T_3. The former spaces are called regular, and the latter normal.

Specifically a space X is said to be regular if and only if it is both a T_0 and a T_3 space, normal if and only if it is both a T_1 and T_4 space, completely normal if and only if it is both a T_1 and a T_5 space. Then we have the following implications:

Completely normal $\Rightarrow$ Normal $\Rightarrow$ Regular $\Rightarrow$ Hausdorff $\Rightarrow T_1 \Rightarrow T_0$

The use of terms 'regular' and 'normal' is not uniform throughout the literature. While some authors use these terms interchangeably with 'T_3 space' and 'T_4 space' respectively, others refer to our T_3 space as a 'regular' space and vice versa, and similarly permute 'T_4 space' and 'normal'. This allows the successively stronger properties to correspond to increasing T_i axioms.

12.3.1 Hausdorff Spaces

Theorem 4: Let $(X, \mathcal{T})$ be a topological space. Then the following statements are equivalent:

1. (T_2 axiom). Any two distinct points of X have disjoint neighbourhoods.
2. The intersection of the closed neighborhoods of any point of X consists of that point alone.
3. The diagonal of the product space $X \times X$ is a closed set.
4. For every set I, the diagonal of the product space $Y + X^I$ is close in Y.
5. No filter on X has more than one limit point.
6. If a filter $\mathcal{F}$ on X converges to x, then x is the only cluster point of $\mathcal{F}$.

Proof: We will proof the implications:

(1) $\Rightarrow$ (2) $\Rightarrow$ (6) $\Rightarrow$ (5) $\Rightarrow$ (1)

(1) $\Rightarrow$ (4) $\Rightarrow$ (3) $\Rightarrow$ (1)

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$(1) \Rightarrow (2)$: If $x \neq y$ there is an open neighbourhood U of x and an open neighbourhood V of y such that $U \cap V = \varnothing$; hence $y \notin \overline{U}$.

$(2) \Rightarrow (6)$: Let $x \neq y$; then there is a closed neighbourhood V of x such that $y \notin V$, and by hypothesis there exists $M \in \mathcal{F}$ such that $M \subset V$; thus $M \cap CV = \varnothing$. But CV is a neighbourhood of y; hence y is not a cluster point of $\mathcal{F}$.

$(6) \Rightarrow (5)$: Clear, since every limit point of a filter is also a cluster point.

$(5) \Rightarrow (1)$: Suppose $x \neq y$ and that every neighbourhood V of x meets every neighbourhood W of y. Then the sets $V \cap W$ form a basis of a filter, which has both x and y as limits points, which is contrary to hypothesis.

$(1) \Rightarrow (4)$: Let $(x) = (x_i)$ be a point of X^I which does not belong to the diagonal Δ. Then there are at least two indices λ, μ such that $x_\lambda \neq x_\mu$. Let V_λ (respectively V_μ) be a neighbourhood of x_λ (respectively x_μ) in X, such that $V_\lambda \cap V_\mu = \varnothing$; then the set $W = V_\lambda \times V_\mu \times \prod_{i \neq \lambda, \mu} X_i$ (where $X_i = X$ if $i \neq \lambda, \mu$) is a neighbourhood of x in X^I which does not meet Δ. Hence Δ is closed in X^I.

$(4) \Rightarrow (3)$: Obvious.

$(3) \Rightarrow (1)$: If $x \neq y$ then $(x, y) \in X \times X$ is not in the diagonal Δ, hence there is a neighbourhood V of x and a neighbourhood W of y in X such that $(V \times W) \cap \Delta = \varnothing$, which means that $V \cap W = \varnothing$.

Let $f : X \to Y$ be a mapping of a set X into a Hausdorff space Y; then it follows immediately from Theorem 2.24 that f has at most one limit with respect to a filter $\mathcal{F}$ on X, and that if f has y as a limit with respect to $\mathcal{F}$, then y is the only cluster point of f with respect to $\mathcal{F}$.

Theorem 5: Let f, g be two continuous mappings of a topological space X into a Hausdorff space Y; then the set of all $x \in X$ such that $f(x) = g(x)$ is closed in X.

Corollary 1 (Principle of extension of identities): Let f, g be two continuous mappings of a topological space X into a Hausdorff space Y. If $f(x) = g(x)$ at all points of a dense subset of X, then $f = g$.

In other words, a continuous map of X into Y (Hausdorff) is uniquely determined by its values at all points of a dense subset of X.

Corollary 6: If f is a continuous mapping of a topological space X into a Hausdorff space Y, then the growth of f is closed in $X \times Y$.

For this graph is the set of all $(x, y) \in X \times Y$ such that $f(x) = y$ and the two mappings $(x, y) \to f(x)$ are continuous.

The invariance properties of Hausdorff topologies are:

Theorem 7:

1. Hausdorff topologies are invariant under closed bijections.
2. Each subspace of a Hausdorff space is also a Hausdorff space.
3. The Cartesian product $\prod\{X_\alpha \mid \alpha \in A\}$ is Hausdorff if and only if each X_α is Hausdorff.

Proof:

1. Since a closed bijection is also an open map, the images of disjoint neighborhoods are disjoint neighbourhoods and the result follows.
2. Let $A \subset X$ and $p, q \in A$; since there are disjoint neighbourhoods $U(p)$, $U(q)$ in X, the neighbourhoods $U(p) \cap A$ and $U(q) \cap A$ in A are also disjoint.
3. Assume that each X_α is Hausdorff and that $\{p_\alpha\} \neq \{q_\alpha\}$; then $p_\alpha \neq q_\alpha$ for some α. So choosing the disjoint neighbourhoods $U(p_\alpha)$, $U(q_\alpha)$ gives the required disjoint neighbourhoods $(U(p_\alpha))$, $(U(q_\alpha))$ in $\prod_\alpha X_\alpha$. Conversely, if $\prod_\alpha X_\alpha$ is Hausdorff, then each X_α is homeomorphic to some slice in $\prod_\alpha X_\alpha$, so by (2) (since the Hausdorff property is a topological invariant), X_α is Hausdorff.

Theorem 8: If every point of a topological space X has a closed neighbourhood, which is a Hausdorff subspace of X, then X is Hausdorff.

12.3.2 Convergence to T_0 Space

Definition : Let (X, T) be a topological space and $x \in X$. Let $<x_n>$ be a sequence of points in X. Then the sequence has limit x or converges to x written as $\lim x_n = x$ or $x_n \to x$ if and only if for every open set G containing x there exists an integer $N(G)$ such that $x_n \in G$ whenever $n > N(G)$.

A sequence will be called convergent if and only if there is at least one point to which it converges. Every subsequence of a convergent sequence is also convergent and has the same limits. The convergence of a sequence and its limits are not affected by a finite number of alternations in the sequence, including the

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adding or removing of a finite number of terms of the sequence. It is the failure of limits of sequences to be unique that makes this concept unsatisfactory in general topological spaces. For example let T be the trivial topology on a set X and let $<x_n>$ be a sequence in X and x be any member of X, then $<x_n> \to x$.

However in a Hausdorff space, a convergent sequence has a unique limit as the following theorem shows.

Theorem 9: In a Hausdorff space, a convergent sequence has a unique limit.

Proof: Suppose a sequence $<x_n>$ converges to two distinct points x and x^* in a Hausdorff space X. By the Hausdorff property, there exists two disjoint open sets G and G^* such that $x \in G$ and $x^* \in G^*$. Since $x_n \to x$, there exists an integer N such that $x_n \in G$ whenever $n > N$. Also $x_n \to x^*$, there exists an integer N such that $x_n \in G^*$ whenever $n > N^*$. If m is any integer greater than both N and N^*, then x_m must be in both G and G^* which contradicts that G and G^* are disjoint.

Note: The converse of this theorem is not true.

Theorem 10: If $<x_n>$ is a sequence of distinct points of a subset E of a topological space X which converges to a point $x \in X$ then x is a limit point of the set E.

Proof: If x belongs to the open set G, then there exists an integer $N(G)$ such that, $x_n \in G$ for all $n > N(G)$. Since the points x_n's are distinct, at most one of them equals x and so $E \cap G - \{x\} \neq \phi$; this implies that x is a limit point of E.

Note: The converse of this theorem is not true, even in a Hausdorff space.

Theorem 11: If f is a continuous mapping of the topological space X into the topological space X^*, and $<x_n>$ is a sequence of points of X, which converges to the point $x \in X$, then the sequence $<f(x_n)>$ converges to the point $f(x) \in X^*$.

Proof: If $f(x)$ belongs to the open set G^* in X^*, then $f^{-1}(G^*)$ is an open set in X containing x, since f is continuous and we know that if f is continuous, then inverse image of every open set is open, then there must exist an integer N such that $x_n \in f^{-1}(G^*)$, whenever $n > N$. Thus we have $f(x_n) \in G^*$ when $n > N$ and so $f(x_n) \to f(x)$. Hence $<f(x_n)>$ converges to the point $f(x) \in X^*$.

Note: The converse of this theorem is also not true even in a Hausdorff space, that is, the mapping f for which $x_n \to x$ implies $f(x_n) \to f(x)$, may be continuous. For example, let X be the Hausdorff space of ordinals less than or equal to the first uncountable ordinal r with order topology. The real valued function f, defined by setting $f(\alpha) = 0$ if $\alpha < r$ and $f(r) = 1$ is not continuous at r, even though it does preserve convergent sequences.

Theorem 12: An infinite Hausdorff space X contains an infinite sequence of non-empty disjoint open sets.

Proof: If X has no limit point, then X must have the discrete topology since singletons are closed in a Hausdorff space. Thus any infinite sequence of distinct points of X would serve as desired sequence.

Suppose, then that x is a limit point of X. Choose x_1 to be any point of X different from x. Since X is Hausdorff, there exists two disjoint open sets G_1 and V_1 such that $x_1 \in G_1$ and $x \in V_1$. Since x is a limit point of X belonging to the open set V_1, there exists some point $x_2 \in X \cap V_1 - \{x\}$. Again since X is Hausdorff, there exist two disjoint open sets G_2* and V_2* such that $x_2 \in G_2$*and $x \in V_2$*. If we let $G_2 = G_2^* \cap V_1$ and $V_2 = V_2^* \cap V_1$, then G_2 and V_2 are disjoint open sets contained in V_1 and hence disjoint from G_1 containing x_2 and x respectively.

We will now proceed by using an inductive argument. Since we have already defined the points $\{x_k\}$ and the open sets $\{G_k\}$ and $\{V_k\}$ with the properties that $x_k \in G_k \subseteq V_{k-1}$, $x \in V_k \subseteq V_{k-1}$ and $G_k \cap V_k = \phi$ for all $k \leq n$. Now x is a limit point of X belonging to the open set V_n and so there exists some point $x_{n+1} \in X \cap V_n - \{x\}$. Since X is Hausdorff, there exist two disjoint open sets G^*_{n+1} and V^*_{n+1} such that $x_{n+1} \in G^*_{n+1}$ and $x \in V^*_{n+1}$. If we let $G_{n+1} = G^*_{n+1} \cap V_n$ and $V_{n+1} = V^*_{n+1} \cap V_n$, then G_{n+1} and V_{n+1} are two disjoint open sets contained in V_n (and hence disjoint from G_n) containing x_{n+1} and x respectively. Since the sets $\{V_n\}$ are monotonic decreasing, we see that G_{n+1} is not only disjoint from G_n but is also disjoint from G_k for $k \leq n$. Since $x_n \in G_n$, the infinite sequence $< G_n >$ defined by induction is the desired sequence of nonempty, disjoint open sets.

12.3.3 First and Second Countable Spaces

First Countable Space

A topological space X is called a first countable space if it satisfies the following axiom, called the first axiom of **countability**:

For each point $p \in X$ there exists a countable class $\mathbf{B_p}$ of open sets containing p such that every open set G containing p also contains a member of $\mathbf{B_p}$. In other words, a topological space X is a first countable space if and only if there exists a countable local base at every point $p \in X$.

A first countable, separable Hausdorff space (in particular, a separable metric space) has at most the continuum cardinality c. In such a space, closure is determined by limits of sequences and any sequence has at most one limit, so there is a surjective map from the set of convergent sequences with values in the countable dense subset to the points of X. A separable Hausdorff space has cardinality at most 2^c where c is the cardinality of the continuum. For this closure is characterized in terms of limits of filter bases: if Y is a subset of X and z is a point of X, then z is in the closure of Y if and only if there exists a filter base B consisting of subsets of Y which converges to z. The cardinality of the set $S(Y)$ of such filter bases is at most $2^{2^{[Y]}}$. Moreover, in a Hausdorff space, there is at most one limit to every filter base. Therefore, there is a surjection $S(Y) \to X$ when $\overline{Y} = X$. The same arguments establish a more general result: suppose that a Hausdorff topological space X contains a dense subset of cardinality k. Then X has cardinality at most 2^{2^k} and cardinality at most 2^k if it is first countable. The product of at most continuum many separable spaces is a separable space. In particular the space R^R of all functions from the real line to itself, endowed with the product topology, is a separable Hausdorff space of cardinality 2^c. More generally, if k is any infinite cardinal, then a product of at most 2^k spaces with dense subsets of size at most k has itself a dense subset of size atmost k.

Example 1: Let X be a metric space. Let $p \in X$. The countable class of open spheres $\{S(p, 1), S(p, 1/2), S(p, 1/3),...\}$ with center at p is a local base at p. Thus every metric space satisfies the first axiom of countability.

Example 2: Let X be any discrete space and let $p \in X$. A local base at point p is the singleton set $\{p\}$ — which is countable. Thus every discrete space satisfies the first axiom of countability.

The first uncountable ordinal ω_1 in its order topology is not separable.

The Banach space l^∞ of all bounded real sequences, with the supremum norm, is not separable.

Second Countable Space

A topological space X with topology τ is called a **second countable space** if it satisfies the following axiom, called the second axiom of **countability.** There exists a countable base **B** for the topology τ. For example, the class of open intervals (a, b) with rational end points — i.e., $a, b \in Q$ where Q is the set of rational numbers — is countable and is a base for the usual topology on the real line R. Thus R satisfies the second axiom of countability and is thus a second countable space.

Consider the real line R with the discrete topology **D**. Now the one and only base for a discrete topology on a set X is that collection B of all the singleton sets of X. We now note that R is non-countable and thus the class of singleton sets $\{p\}$ of R is non-countable. Thus the topological space R with the discrete topology **D** does not satisfy second axiom of countability. By the same logic, the topological space Q of rational numbers with the discrete topology **D** does satisfy second axiom of countability.

If **B** is a countable base for a space X, and if $\mathbf{B}_p$ consists of the members of **B** which contain the point $p \in X$, then $\mathbf{B}_p$ is a countable local base at p Any second-countable space is separable: if $\{U_n\}$ is a countable basis, choosing any $x_n \in U_n$ gives a countable dense subset. Conversely, a metrizable space is separable if and only if it is second countable if and only if it is Lindelöf.

An arbitrary subspace of a second countable space is second countable; subspaces of separable spaces need not be separable.

A product of at most continuum many separable spaces is separable. A countable product of second countable spaces is second countable, but an uncountable product of second countable spaces need not even be first countable.

Any continuous image of a separable space is separable, even a quotient of a second countable space need not be second countable.

The property of separability does not in and of itself give any limitations on the cardinality of a topological space: any set endowed with the trivial topology is separable, as well as second countable, quasi-compact and connected. The 'trouble' with the trivial topology is its poor separation properties: its Kolmogorov quotient is the one-point space.

Theorem 13: A function defined on a first countable space X is continuous at $p \in X$ if and only if it is sequentially continuous at p.

In other words, if a topological space X satisfies the first axiom of countability, then $f: X \to Y$ is continuous at $p \in X$ if and only if for every sequence $\{a_n\}$ in X converging to p, the sequence $\{f(a_n)\}$ in Y converges to $f(p)$, i.e.,

$$a_n p \Rightarrow f(a_n) \to f(p)$$

Theorem 14: A second countable space is also first countable.

Let S be a set. Let A be a subset of S. Then a collection C of subsets of S is a **cover of** A if A is a subset of the union of the members of C, i.e.,

$$A \subset \bigcup \{c : c \in C\}$$

If each member of C is an open subset of S, then C is called an **open cover** of A. If C contains a countable subclass which also is a cover of A, then C is said to be **reducible to a countable cover** of A.

Theorem 15: Let A be any subset of a second countable space X. Then every open cover of A is reducible to a countable cover.

Theorem 16: Let X be a second countable space. Then every base **B** for X is reducible to a countable base for X.

Lindelof space: A topological space X is called a Lindelof space if every open cover of X is reducible to a countable cover.

Thus, every second countable space is a Lindelof space.

12.3.4 Lindelof's Theorems

Let $C=\{A_\lambda : \lambda \in A\}$ be a collection of sets of real numbers. We say that C is a cover (or covering) of a set A of real numbers if each point of A belongs to A_λ for same $\lambda \in A$, that is, if $A \subset \bigcup \{A_\lambda : \lambda \in A\}$. If C is a collection of open sets, then C is called an open covering of A.

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If C' is a cover of A such that $C' \subset C$, is called a subcover of A.

A set A is said to be compact if each open cover of A has a finite subcover.

Example 3: Let $C = \{]-n, n[: n \in N\}$

and $C' = \{]-3n, 3n[: n \in N\}$.

Then C and C' are both open covers of R. Also C' is a subcover of C.

Theorem 17 (Lindelof): Let C be a collection of open sets of real numbers. Then there exists a countable subcollection $\{G_1\}$ of C such that

$$\cup \{G : G \in C\} = \bigcup_{n=1}^{\infty} G_{i.}.$$

Proof: Let $S = \cup \{G : G \in C\}$ and let $x \in S$. Then there is at least one $G \in C$ such that $x \in$ G. Since G is open, there exists an open interval I_x with centre at x such that $I_x \subset G$. Since every interval I_x contains infinitely many rational points we can find an open interval J_x with rational end points such that $x \in J_x \subset I_x$. Since the collection $\{J_x\}$, $x \in S$, is countable, and

$$S = \cup \{J_x : x \in S\}.$$

For each interval in $\{J_x\}$, choose a set G in C which contains it. This gives a countable subcollection

$$\{G_i\}_{i=1}^{\infty} = \text{of C and } S = \bigcup_{i=1}^{\infty} G_i$$

Hence $\cup\{G : G \in C\} = S = \bigcup_{i=1}^{\infty} G_i$.

Theorem 18 (Lindelof covering theorem): Let A be a set of real numbers and let C be an open cover of A. Then there exists a countable subcollection of C which also covers A.

Proof: Since C is an open cover of A, we have

$$A \subset \cup\{G : G \in C\}.$$

Now show that there exists a countable subcollection $\{G_i\}_{i=1}^{\infty}$ of C such that

$$\cup\{G : G \in C\} = S = \bigcup_{i=1}^{\infty} G_i$$

and so $A = \bigcup_{i=1}^{\infty} G_i$ Hence the theorem.

The Heine-Borel covering theorem states that from any open covering of an arbitrary set A of real numbers, we can extract a countable covering. The Heine-Borel theorem tells us that if, in addition, we know that A is closed and bounded, we can reduce the covering to a finite covering. The proof makes use of the nested interval theorem.

Theorem 19 (Heine-Borel): Let F be a closed and bounded set of real numbers. Then each open covering of F has a finite subcovering. That is, if C is a collection of open sets such that $F \subset \cup\{G : G \in C\}$, then there exists a finite subcollection $(G_1\ G_2,...,G_n)$ of C such that $F \subset \bigcup_{i=1}^{n} G_i$.

In other words, every closed and bounded set of real numbers is compact.

Proof: Since F is bounded, it is contained in some closed and bounded interval $[a, b]$. Let C^* be the collection obtained by adding F' (the complement of F) to C; that is, $C^* = C \cup \{F'\}$. Since F is closed, F' is open and so C' is a collection of open sets. By hypothesis, $F \subset \cup\{G : G \in C\}$, and so

$$R = F \cup F' \subset [\cup\{G : G \in C\}] \cup F' = \cup\{G : G \in C^*\}$$

We now show that $[a, b]$ is covered by a finite sub-collection of C. Suppose the contrary to be true. Now if $I_0 = [a, b]$ cannot be covered by a finite subcollection of C^*, then one of the intervals $[a, \frac{1}{2}(a + b)]$, $[\frac{1}{2}(a + b), b]$ cannot be so covered. We rename such an interval as $[a_1, b_2]$. Similarly, one of the intervals $\left[a_1, \frac{1}{2}(a_1 + b_1)\right], \left[\frac{1}{2}(a_1 + b_1), b_1\right]$ cannot be covered by a finite subcollection of C^*. We designate such an interval as $[a_2, b_2]$. Continuing in this way, we obtain a sequence of closed intervals

$$\{I_n\} = \{[a_n, b_n]\} \text{ such that } I_n \subset I_{n+1}$$

and $$|I_n| = b_n - a_n = \frac{b-a}{2^n} \to C \text{ as } n \to \infty.$$

Hence by the nested interval theorem, $\bigcup_{n=1}^{\infty} I_n$ consists of a single point, say x_0. This point x_0 must belong to one of the members, say G_0 of C^*. Since G_0 is an open set and $x_0 \in G_0$, there exists $\varepsilon > 0$ such that $]x_0 - \varepsilon, x_0 + \varepsilon[\subset G_0$. Also we can choose k so large that $\frac{b-a}{2^k} < \varepsilon$. Then $I_k \subset]x_0 - \varepsilon, x_0 + \varepsilon[\subset G_0$ and so I_k is covered by a single member of C^*. But this

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contradicts the constructed property of the members of $\{I_n : n \in \mathrm{N}\}$. Hence $[a, b]$ must be covered by a finite subcollection of C^* and hence F must also be covered by a finite subcollection of C^*. If this finite subcollection does not contain F', it is a subcollection of C and the conclusion of our theorem holds. If the subcollection contains F', denote it by $\{G_1\ G_2 ..., G_n,\ F'\}$. Then $F \subset F' \cup G_1 \cup G_2 \cup ... \cup G_n$. Since no point of F is contained in F', we have $F \subset G_1 \cup G_2 \cup ..... \cup G_n$, and the collection $\{\mathrm{G}_1, G_2, ..., G_n\}$ is a finite subcollection of C which covers F.

This proves the theorem.

The converse of the preceding theorem is given in the next theorem.

Theorem 20: Compact subsets of R are closed and bounded.

Proof: Let A be any compact subset of R. If $A_n =]-n, n[$, then the collection $C = \{A_n : n \in N\}$ is evidently an open cover of R and so an open cover of A. Since A is compact, there exist finitely many positive integers $n_1, n_2 ... n_k$ such that the subcollection $\left\{A_{n_1}, A_{n_2}, ..., A_{n_k}\right\}$ of C covers A. Let $n_0 = \max\ \{n_1, n_2 ..., n_k\}$, then evidently $A \subset A_{n_0} =]-n_0, n_0[$. This implies that A is bounded.

If we can show that no point of $R - A$ can be a limit point of A, then A will be closed. So let $a \in \mathrm{R}$. Then $a \notin A$. Consider the family of closed sets $F_n = [a - 1/n, a + 1]$ for each $n \in N$. Then $C' = \{R - F_n : n \in N\}$ is a collection of open sets. Also evidently $\bigcap_{n=1}^{\infty} F_n = \{a\}$. Since $a \notin A$, we have,

$$A \subset [R - \cap\{F_n : n \in N\}] = \cup\{R - F_n : n \in N\} \text{ [De Morgan law]}.$$

Thus A is covered by the collection C. Hence by compactness of A, there exists finitely many positive integers $m_1; m_2, ..., m_s$ such that every point of A is contained in one of the open sets $R - F_{m_1}, R - F_{m_2}, ..., R - F_{m_s}$. Hence if $x \in A$, then for some $i \in \{1, 2, 3, ..., s\}, x \in \mathrm{R} - F_{m_i}$, which implies that no point of A is contained in $F_{m_i} - [a - 1/m_i, a + 1/m_i]$. This implies that a is not a limit point of A. Hence A is closed.

A Characterization of Compact Sets on R

Theorem 21: A subset of R is compact if and only if it is closed and bounded.

Theorem 22: R is not compact.

Proof: Let $A_n =]-n, n[$. Then clearly $C = \{A_n : n \in N\}$ is an open cover of R. If $\{A_{n_1}, A_{n_2}, ..., A_{n_k}\}$ be any finite subfamily of C, let $n_0 = \max\ \{n_1, n_2,...,n_k\}$. Then $n_0 \notin A_{n_i}$, for any $i = 1, 2,...,k$.

It follows that no finite subfamily of C can cover R. Hence R is not compact.

Theorem 23: Show that open intervals on 1 are not compact.

Proof: Let $I =]a, b[$ be any open interval on R.

If $A_n =]a + 1/n, b[$, then evidently the collection $C = \{A_n : n \in N\}$ is an open cover of $]a, b[$ since $\bigcup_{n=1}^{\infty} A_n =]a, b[$. But it is not possible to find a finite subcollection of C which covers A. For, if $C' = \{A_{n_1}, A_{n_2}, ..., A_{n_k}\}$ be any finite subcollection of C, let $n_0 = \max\ \{n_1\ n_2, ..., n_k\}$. Then it is evident that the subset $]a, a + 1/n_0]$ of A is not covered by C'. Thus we have shown that there exists an open cover of A which does not admit of a finite subcover. Hence $]a, b[$ is not compact.

12.3.5 Separable Spaces and Separability

A topological space is called **separable** if it contains a countable dense subset; that is, there exists a sequence $\{x_n\}_{n=1}^{\infty}$ of elements of the space such that every nonempty open subset of the space contains at least one element of the sequence. In particular, every continuous function on a separable space whose image is a subset of a Hausdorff space is determined by its values on the countable dense subset.

In general, separability is a technical hypothesis on a space which is quite useful and among the classes of spaces studied in geometry and classical analysis generally considered to be quite mild. It is important to compare separability with the related notion of second countability, which is in general stronger but equivalent on the class of metrizable spaces. Let us consider some examples of separable space.

- Every compact metric space (or metrizable space) is separable.
- Any topological space which is the union of a countable number of separable subspaces is separable. Together, these first two examples give a different proof that n-dimensional Euclidean space is separable.

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- The space of all continuous functions from a compact subset of R^n into R is separable.
- It follows easily from the Weierstrass approximation theorem that the set $Q[t]$ of polynomials with rational coefficients is a countable dense subset of the space $C([0,1])$ of continuous functions on the unit interval [0, 1] with the metric of uniform convergence. The Banach-Mazur theorem asserts that any separable Banch space is isometrically isomorphic to a closed linear subspace of $C([0, 1])$.
- The Lebesuge spaces L^p are separable for any $1 \leq p \leq \infty$.
- A Hilbert space is separable if and only if it has countable orthonormal basis; it follows that any separable, infinite-dimensional Hilbert space is isometric to ℓ^2.
- An example of a separable space that is not second-countable is $R_{llt,}$ the set of real numbers equipped with the lower limit topology.
- A topological space X is separable if and only if there exists a finite or denumerable subset A of X such that the closure of A is the entire space i.e., $\overline{A} = X$.
- The real line R with the usual topology is separable since the set Q of rational numbers is denumerable and is dense in R, i.e., $\overline{Q} = R$.
- Every second countable space is separable but not every separable space is second countable. For example, the real line R with the topology generated by the closed-open intervals $[a, b)$ is a classic example of a separable space which does not satisfy the second axiom of countability.

Check Your Progress

3. Define the principle of extension of identities.
4. What do you understand by a convergent sequence?
5. What is meant by the Lindel of covering theorem?
6. What is a separable topological space?

12.4 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1. A topological space X is said to satisfy the First Axiom of Countability if, for every $x \in X$ there exists a countable collection $\mathcal{U}$ of neighbourhoods of x, such that if N is any neighbourhood of x, then there exists $U \in \mathcal{U}$ with $U \subseteq N$.

2. A topological space that satisfies the first axiom of countability is said to be First Countable.
3. Let f, g be two continuous mappings of a topological space X into a Hausdorff space Y. If $f(x) = g(x)$ at all points of a dense subset of X, then $f = g$.
4. A sequence will be called convergent if and only if there is at least one point to which it converges. Every subsequence of a convergent sequence is also convergent and has the same limits.
5. Let A be a set of real numbers and let C be an open cover of A. Then there exists a countable subcollection of C which also covers A.
6. A topological space is called separable if it contains a countable dense subset; that is, there exists a sequence $\{x_n\}_{n=1}^{\infty}$ of elements of the space such that every nonempty open subset of the space contains at least one element of the sequence.

12.5 SUMMARY

- The separation axioms T_i specify the degree to which distinct points or closed sets may be separated by open sets.
- Let (X, T) be a topological space. Then the following statements are equivalent:
  - o (T_2 axiom). Any two distinct points of X have disjoint neighbourhoods.
  - o The intersection of the closed neighborhoods of any point of X consists of that point alone.
  - o The diagonal of the product space $X \times X$ is a closed set.
  - o For every set I, the diagonal of the product space $Y + X^I$ is close in Y.
  - o No filter on X has more than one limit point.
  - o If a filter $\mathcal{F}$ on X converges to x, then x is the only cluster point of $\mathcal{F}$.
- Let f, g be two continuous mappings of a topological space X into a Hausdorff space Y; then the set of all $x \in X$ such that $f(x) = g(x)$ is closed in X.
- If f is a continuous mapping of a topological space X into a Hausdorff space Y, then the growth of f is closed in $X \times Y$.
- If every point of a topological space X has a closed neighbourhood, which is a Hausdorff subspace of X, then X is Hausdorff.

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- The convergence of a sequence and its limits are not affected by a finite number of alternations in the sequence, including the adding or removing of a finite number of terms of the sequence.
- In a Hausdorff space, a convergent sequence has a unique limit.
- If $<x_n>$ is a sequence of distinct points of a subset E of a topological space X which converges to a point $x \in X$ then x is a limit point of the set E.
- If f is a continuous mapping of the topological space X into the topological space X^*, and $<x_n>$ is a sequence of points of X, which converges to the point $x \in X$, then the sequence $<f(x_n)>$ converges to the point $f(x) \in X^*$.
- An infinite Hausdorff space X contains an infinite sequence of non-empty disjoint open sets.
- A function defined on a first countable space X is continuous at $p \in X$ if and only if it is sequentially continuous at p.
- A second countable space is also first countable.
- Let A be any subset of a second countable space X. Then every open cover of A is reducible to a countable cover.
- Let X be a second countable space. Then every base **B** for X is reducible to a countable base for X.

12.6 KEY WORDS

- **Separation axioms:** Separation axioms specify the degree to which distinct points or closed sets may be separated by open sets.
- **Hausdorff space**: It is a topological space in which each pair of distinct points can be separated by a disjoint open set.
- **First countable space**: A topological space X is called a first countable space if it satisfies the first axiom of countability.
- **Lindelof space:** A topological space X is called a Lindelof space if every open cover of X is reducible to a countable cover.

12.7 SELF ASSESSMENT QUESTIONS AND EXERCISES

Short Answer Questions

1. Write a short note on the axioms of countability.
2. Discuss convergence to T_0 space.

3. Describe Lindelof covering theorem.
4. What do you understand by characterisation of compact sets on R?

Long Answer Questions

1. Give a detailed account of the separation axioms.
2. Describe first and second countable spaces.
3. Explain separable spaces and separability.

12.8 FURTHER READINGS

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UNIT 13 NORMAL SPACES

NOTES

Structure

13.0 INTRODUCTION

A topological space X is a normal space if, given any disjoint closed sets E and F, there are neighbourhoods U of E and V of F that are also disjoint.This condition says that E and F can be separated by neighbourhoods. Urysohn's lemma states that a topological space is normal if and only if any two disjoint closed subsets can be separated by a continuous function. Urysohn's lemma is commonly used to construct continuous functions with various properties on normal spaces. It is widely applicable since all metric spaces and all compact Hausdorff spaces are normal. The lemma is generalized by (and usually used in the proof of) the Tietze extension theorem. The lemma is named after the mathematician Pavel Samuilovich Urysohn. In this unit, you will learn about normal spaces. A regular space is a topological space in which every neighbourhood of a point contains a closed neighbourhood of the same point. You will discuss the Urysohn's lemma and Tietze extension theorem. You will understand complete regularity and normality.

13.1 OBJECTIVES

After going through this unit, you will be able to:

- Understand normal spaces
- Discuss the Urysohn's lemma
- Describe Tietze extension theorem
- Interpret the Urysohn's lemma and Tietze extension theorem
- Learn complete regularity and normality

13.2 NORMAL SPACES

NOTES

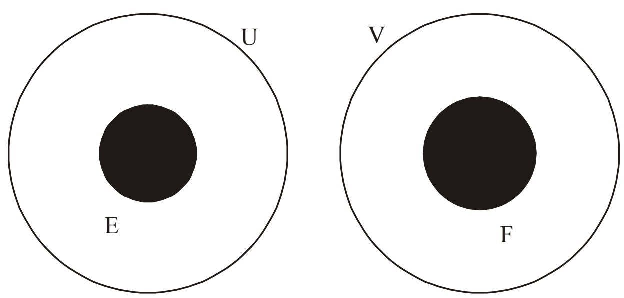
A topological space X is a **regular space** if, given any nonempty closed set F and any point x, such that $x \notin F$, there exists a neighbourhood U of x and a neighbourhood V of F that are disjoint or we can say that it must be possible to separate x and F with disjoint neighbourhoods.

If a topological space is both regular and a Hausdorff space then it is said to be a $\mathbf{T}_3$ space or regular Hausdorff space. A Hausdorff space or $\mathbf{T}_2$ space is a topological space in which any two distinct points are separated by neighbourhoods. Also, a space is $\mathbf{T}_3$ iff it is both regular and $\mathbf{T}_0$. A $\mathbf{T}_0$ or Kolmogorov space is a topological space, wherein any two distinct points are topologically distinguishable, i.e., for every pair of distinct points, at least one of them has an open neighborhood not containing the other. In fact, if a space is Hausdorff then it is $\mathbf{T}_0$ and each $\mathbf{T}_0$ regular space is Hausdorff, since given two distinct points, at least one of them misses the closure of the other one. So by regularity there exist disjoint neighbourhoods separating one point from the closure of the other.

A topological space is said to be topologically regular if every point has an open neighbourhood that is regular. Clearly, every regular space is locally regular, but the converse is not true.

Normal Spaces

A topological space X is said to be a **normal space** if, given any disjoint closed sets E and F, there are open neighbourhoods U of E and V of F that are also disjoint. This condition implies that E and F can be separated by neighbourhoods.



The closed sets E and F, represented in the Figure by closed disks on opposite sides of the picture, are separated by their respective neighbourhoods U and V, represented by larger, but still disjoint open disks.

A $\mathbf{T}_4$ space is a normal $\mathbf{T}_1$ space X. This is equivalent to X being Hausdorff and normal. A **completely normal space** or a **hereditarily normal space** is a topological space X in which every subspace of X with subspace topology is a normal space. It turns out that X is completely normal if and only if every two separated sets can be separated by neighbourhoods. A **completely T_4 space** or **T_5 space** is a completely normal Hausdorff topological space X, or every subspace of X must be a $\mathbf{T_4}$ space.

A **perfectly normal space** is a topological space X in which every two disjoint non-empty closed sets E and F can be precisely separated by a continuous function f from X to the real line **R**. The preimages of $\{0\}$ and $\{1\}$ under f are, respectively, E and F. In this definition, the real line can be replaced with the unit interval [0, 1].

A locally normal space is a topological space where every point has an open neighbourhood that is normal. Every normal space is locally normal, but the converse is not true.

Check Your Progress

1. Define a regular space.
2. What is a normal space?
3. What is a complete normal space?
4. What do you mean by a perfectly normal space?

13.3 THE URYSOHN'S LEMMA

Theorem 1 (Urysohn's Lemma): Let X be a topological space and let any two disjoint closed sets A, B in X can be separated by open neighbourhoods. Then there is a continuous function $f : X \to [0, 1]$ such that $f \mid A \equiv 1$ and $f \mid B \equiv 0$.

Proof: Here, define f as the pointwise limit of a sequence of functions. Any collection of sets $\mathcal{U}_r = (A_0, A_1, \ldots, A_r)$ is an admissible chain if $A = A_0 \subset A_1 \subset, ...,$ $\subset A_r \subset X \backslash B$ and $\overline{A}_{k-1} \subset \overset{0}{A}_k, 0 \le k \le r$. The set $\overset{0}{A}_{k+1} \setminus \overline{A}_{k-1}$ is the kth step domain of U_r, where $A_{r+1} = X$ and $A_{-1} = \phi$.

Fig. 3.1 *An Admissible Chain*

Each pair of adjacent shaded regions represents a step domain. We will prove this theorem with the help of a series of lemmas.

Lemma 1: Each $x \in X$ lies in some step domain for any U_r.

Proof: Take $x \in X$ and any admissible chain U_r. Let k, $0 \le k \le r+1$, be the smallest number such that $x \in \overset{0}{A}_k$. Then $x \in \overset{0}{A}_k \setminus \overline{A}_{k-2}$.

Lemma 2: Each step domain is open.

Proof: Since $\overset{0}{A}_{k+1} \setminus \overline{A}_{k-1} = \overset{0}{A}_{k+1} \cap \left(X \setminus \overline{A}_{k-1}\right)$ is the finite intersection of open sets, hence it is open. For any U_r, define the uniform step function $f_r : X \to [0, 1]$ as, $f_r|A \equiv 1$, $f_r|(X \setminus A_r) \equiv 0$ and $f_r|(A_k \setminus A_{k-1}) \equiv 1-k/r$, $1 \le k \le r$

Lemma 3: If x and y are in the same step domain, then $|f_r(x) - f_r(y)| \le 1/r$.

Proof: Let $x, y \in \overset{0}{A}_{k+1} \setminus \overline{A}_{k-1}$. If both x and y are in $\overset{0}{A}_{k+1}$ or A_k, then by definition of f_r, $f_r(x) = f_r(y)$. Hence $|f_r(x) - f_r(y)| = 0$. If $x \in \overset{0}{A}_{k+1}$ and $y \in A_k$, then $f_r(x) = 1-(k+1)/r$ and $f_r(y) = 1 - k/r$. So $|f_r(x) - f_r(y)| = 1/r$.

The above three lemmas will be used in the last step of the proof.

Now, let the admissible chain $\mathcal{U}_{2r-1} = (A_0, A'_1, \ldots, A'_r, A_r)$ be a refinement of the admissible chain $\mathcal{U}_r = (A_0, A_1, \ldots, A_r)$.

We can say that, the refinement $\mathcal{U}_{2r-1}$ of the admissible chain $\mathcal{U}_r$ contains every set in $\mathcal{U}_r$, and for every $i \ge 1$ contains a set A'_i such that $A_{i-1} \subset A'_i \subset A_i$. Naturally, refinements place new sets between each pair of sets in the original admissible chain.

Lemma 4: Every admissible chain has a refinement.

Proof: It is sufficient to show that for any subsets M, N of X, with $\overline{M} \subset \overset{0}{N}$, there exists $L \subset X$ with $\overline{M} \subset \overset{0}{L} \subset \overline{L} \subset \overset{0}{N}$. Now since $\overline{M} \subset \overset{0}{N}$, $\overline{M} \cap \left(X \setminus \overset{0}{N} = \varphi\right)$ and since $(X \setminus \overset{0}{N})$ is the complement of an open set and therefore closed, there exist disjoint open sets U, V, with $\overline{M} \subset U$ and $(X \setminus \overset{0}{N}) \subset V$. Again since U and V are disjoint, $U \subset (X \setminus V)$. Also because $(X \setminus V)$ is closed and $\overline{U}$ is contained in every closed set containing U, so $\overline{U} \subset (X \setminus V)$. Moreover, $(X \setminus \overset{0}{N}) \subset V$ means $(X$

$\setminus V) \subset \overset{0}{N}$. Putting all this collectively gives, $\overline{M} \subset \overset{0}{U} \subset \overline{U} \subset (X \setminus V) \subset \overset{0}{N}$. Let $L = U$ and the proof is complete.

NOTES

Lemma 5: Let $\mathcal{U}_r$ be an admissible chain with $r + 1$ elements and $\mathcal{U}_s$ be a refinement with $2r+1$ elements. Then $|f_r(x) - f_s(x)| \leq 1/(2r)$.

Proof: Suppose $x \in A_k \setminus A_{k-1}$, where $A_k, A_{k-1} \in U_r$. Then, $f_r(x) = 1 - k/r$. Also, $\mathcal{U}_s = (A_0, A'_1, A_1, \ldots, A'_j, A_j, \ldots, A'_k, A_k) = (A_0, A_{1'}, A_{2'}, \ldots, A_{(2k-1)'}, A_{2k}, A_{(2r-1)'}, A_{2r'})$. Now, x is either in $A_k \setminus A'_k = A_{2k} \setminus A_{(2k-1)}$ or in $A'_k \setminus A_{k-1} = A_{(2k-1)}$. Therefore either, $f_s(x) = 1 - (2k)/(2r) = f_r(x)$ or, $f_s(x) = 1-(2k-1)/(2r) = f_r(x) + 1/(2r)$. Either way we get the desired result.

Now we will define the sequence. Let $\mathcal{U}_0 = (A, X \setminus B)$ and let $\mathcal{U}_{n+1}$ be a refinement of $\mathcal{U}_n$.

By Lemma 4, every admissible chain has a refinement. So we get a sequence of admissible chains. Let f_n be the uniform step function on the nth admissible chain.

Lemma 6: The sequence $\{f_n(x)\}$ converges for each $x \in X$.

Proof: From the definition of the uniform step functions, the sequence is bounded above by 1. Now it remains to prove that the sequence is non-decreasing. Note first that $\mathcal{U}_0$ contains one term excluding A itself and therefore by definition of refinement, $\mathcal{U}_1$ will contain 2 terms excluding A.

Applying induction, $\mathcal{U}_n$ contains 2^n terms excluding A. Notice that for $x \notin A_j \setminus A_{j-1}$ $\forall$ $A_j, A_{j-1} \in \mathcal{U}_r, f_k(x)$ is either 0 or 1, i.e., constant and constant sequences converge. Suppose $x \in A_j \setminus A_{j-1}$, where $A_j, A_{j-1} \in U_r$. Then $f_k(x) = 1-j/k$. Furthermore, $\mathcal{U}_{k+1} = (A_0, A'_1, A_1, \ldots, A'_j, A_j, \ldots, A'_k, A_k) = (A_0, A_{1'}, A_{2'}, \ldots, A_{(2j-1)}, A_{2j}, \ldots, A_{(2k-1)'}, A_{2k'})$ and x is either in $A_j \setminus A'_j = A_{2j} \setminus A_{(2j-1)'}$ or in $A'_j \setminus A_{j-1} = A_{(2j-1)'} \setminus A_{(2j-2)'}$. This implies,

$$f_{k+1}(x) = 1-(2j)\backslash(2k) = f_k(x)$$

or

$$f_{k+1} = 1-(2j-1)\backslash(2k) \geq f_k(x)$$

This proves that the sequence is nondecreasing and hence convergent, because bounded monotonic sequences converge.

For each x, let $f(x) = \lim_{n\to\infty} f_n(x)$. Because each f_n is constantly 1 on A and 0 on B, f will also have this property. For proving that f is continuous, it is sufficient to show that if we take any $f(x) \in [0, 1]$ and any open set $(a, b) \subset [0, 1]$ containing

$f(x)$, then there is an open set $U \subset X$ such that $x \in U$ and $f(U) \subset (f(x) - \varepsilon, f(x) + \varepsilon)$. To prove this, we have to show one more lemma.

NOTES

Lemma 7: For fixed x and any n, $|f(x) - f_n(x)| \leq 1/2^n$.

Proof: $|f(x) - f_n(x)| = |\lim_{\substack{k\to\infty \\ k\geq n}} f_k(x) - f_n(x)|$

$$= |\lim_{\substack{k\to\infty \\ k\geq n}} (f_k(x) - f_n(x))| = \lim_{\substack{k\to\infty \\ k\geq n}} |(f_k(x) - f_{k-1}(x)) +$$

$$(f_{k-1}(x) - f_{k-2}(x)) + (f_{n+1}(x) - f_n(x))|$$

$$\leq \lim_{\substack{k\to\infty \\ k\geq n}} (|f_k(x) - f_{k-1}(x)| + |f_{k-1}(x) - f_{k-2}(x)| + \ldots.. + |f_{n+1}(x) - f_n(x)|)$$

$$\leq \lim_{\substack{k\to\infty \\ k\geq n}} \left(1/2^k + 1/2^{k-1} + \ldots. + 1/2^{n+1}\right)$$

$$= \sum_{k=n+1}^{\infty} 1/2^k = 1/2^n \left(\sum_{k-1}^{\infty} 1/2^k\right) = 1/2^n$$

Take n large enough so that $3/2^n < \varepsilon$, and suppose x lies in the kth step domain,

$S_k = \overset{0}{A}_{k+1} \setminus \overline{A}_{k-1}$ (Since by Lemma 1, every x lies in some step function).

Furthermore, by Lemma 2, this step domain is an open neighbourhood of x. Take any $y \in S_k$. Then by Lemmas 3 and 6,

$$|f(x) - f(y)|$$

$$= |f(x) - f_n(x) + f_n(x) - f_n(y) + f_n(y) - f(y)|$$
$$\leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)|$$

$\leq 1/2^n + 1/2^n + 1/2^n = 3/2^n < \varepsilon$. So every $y \in S_k$ maps into (a, b), which proves that f is continuous.

13.4 TIETZE EXTENSION THEOREM

Before going to main theorem, Lemma 8 is considered.

Lemma 8: If X is a normal topological space and A is closed in X, then for any continuous function $f: A \to R$ such that $|f(x)| \leq 1$, there is a continuous function g:

$X \to R$ such that $|g(x)| \leq \frac{1}{3}$ for $x \in X$, and $|f(x) - g(x)| \leq \frac{2}{3}$ for $X \in A$.

NOTES

Proof: The sets $f^{-1}\left(\left(-\infty - \frac{1}{3}\right]\right)$ and $f^{-1}\left(\left[\frac{1}{3}, \infty\right)\right)$ are disjoint and closed in A. Since A is closed, they are also closed in X. Since X is normal, then by Urysohn's lemma and the fact that [0, 1] is homomorphic to $\left[-\frac{1}{3}, \frac{1}{3}\right]$, there is a continuous function g: $X \to \left[-\frac{1}{3}, \frac{1}{3}\right]$ such that $g\left(f^{-1}\left(-\infty, -\frac{1}{3}c\right]\right) = -\frac{1}{3}$ and $g\left(f^{-1}\left[\frac{1}{3}, \infty\right)\right) = \frac{1}{3}$. Thus $|g(X)| \leq \frac{1}{3}$ for $x \in X$. Now, if $-\infty \leq f(X) \leq -\frac{1}{3}$, then $|g(x)| \leq -\frac{1}{3}$ and thus $|f(x) - g(x)| \leq \frac{2}{3}$. Similarly if $\frac{1}{3} \leq f(x) \leq 1$, then $g(x) = \frac{1}{3}$ and thus $|f(x) - g(x)| \leq \frac{2}{3}$. Finally, for $|f(x)| \leq \frac{1}{3}$ we have that $|g(x)| \leq \frac{1}{3}$, and so $|f(x) - g(x)| \leq \frac{2}{3}$. Hence $|f(x) - g(x)| \leq \frac{2}{3}$ holds for all $x \in A$.

Theorem 2: First suppose that for any continuous function on a closed subset there is a continuous extension. Let C and d be disjoint and closed in X. Define f: $C \cup D \to R$ by $f(x) = 0$ for $x \in C$ and $f(x) = 1$ for $x \in D$. Now f is continuous and we can extend it to a continuous function F: $X \to R$. By Urysohn's lemma, x is normal because F is continuous function such that $F(x) = 0$ for $x \in C$ and $F(x) = 1$ for $x \in D$.

Conversely, let X be normal and A be closed in X. By the Lemma 8, there is a continuous function g_0: $X \to R$ such that $\left|g_0(x) \leq \frac{1}{3}\right|$ for $x \in X$ and $|f(x) - g_0(x)| \leq \frac{1}{3}$ for $x \in A$. Since $(f - g_0)$: $A \to R$ is continuous, the lemma tells us that there is a continuous function g_1: $X \to R$ such that $|g_1(x)| \leq \frac{1}{3}\left(\frac{2}{3}\right)$ for $x \in X$ and $|f(x) - g_0(x) - g_1(x)| \leq \frac{2}{3}\left(\frac{2}{3}\right)$ for $x \in A$.

NOTES

By repeated application of the lemma we can constuct a sequence of continuous function $g_0, g_1, g_2, \ldots..$, such that $|g_n(x)| \le \frac{1}{3}\left(\frac{2}{3}\right)^n$ for all $x \in X$, and

$$|f(x) - g_0(x) - g_1(x) - g_2(x) - \ldots...| \le \left(\frac{2}{3}\right)^n \text{ for } x \in A.$$

Define $F(x) = \sum_{n=0}^{\infty} g_n(x)$. Since $|g_n(x)| \le \frac{1}{3}\left(\frac{2}{3}\right)^n$ and $\sum_{n=0}^{\infty} \frac{1}{3}\left(\frac{2}{3}\right)^n$ converges as a geometric series, then $\sum_{n=0}^{\infty} g_n(x)$ converges absolutely and uniformly, so F is a continuous function defined everywhere. Moreover, $\sum_{n=0}^{\infty} \frac{1}{3}\left(\frac{2}{3}\right)^n =$ 1, implies that $|F(x)| \le 1$.

Now, for $x \in A$, we have that $\left|f(x) \sum_{n=0}^{k} g_n(x)\right| \le \left(\frac{2}{3}\right)^{k+1}$ and as k goes to infinity, the right side goes to zero and so the sum goes to $F(x)$. Thus $|f(x) - F(x)| = 0$ Therefore f extends F.

Notes: If f was a function satisfying $|f(x)| < 1$, then the theorem can be strengthended. Find an extension F of f. The set $B\, F^{-1}(\{-1\} \cup \{1\})$ is closed and disjoint from A because $|F(x)| = |f(x)| < 1$ for $x \in A$. By Urysohn's lemma there is a continuous function f such that $\phi(A) = \{1\}$ and $\phi(B) = \{0\}$. Hence $\phi(x)\phi(x)$ is a continuous extension of $\phi(x)$ and has the property that $|F(x)f(x)| < 1$.

If f is unbounded, then Tietze extension theorem holds as well. To see that consider $t(x) = \tan^{-1}(x)/(n/2)$. The function $t\,.\,f$ has the property that $(t\,.\,f)\,(x) <$ for $x \in a$, and so it can be extended to a continuous function $h: X \to R$ which has the property $|h(x)| < 1$. Hence $t^{-1}\,.\,h$ is a continuous extension of f.

13.4.1 Complete Regularity and Complete Normality

Completely Normal Spaces

Tietze introduced completely normal spaces in 1923.

Definition: A topological space $(X, \mathbf{T})$ is called completely normal if and only if it satisfies the statement given as: if A and B are two separated subsets of X, then there exist two disjoint open sets G and H such that $A \subset G$ and $B \subset H$.

Definition: A $\mathbf{T}_5$ space is a completely normal $\mathbf{T}_1$ space.

Theorem 3: Every completely normal space is normal and consequently every $\mathbf{T}_5$ space is a $\mathbf{T}_4$ space.

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Proof: Suppose $(X, \mathbf{T})$ is a completely normal space and A and B are two closed subsets of X such that $A \cap B = \phi$. Now, as A and B are closed, we have

$$C(A) = A,\ C(B) = B \Rightarrow C(A) \cap B = \phi,\ A \cap C(B) = \phi$$

Hence A and B are separated subsets of X. Now, by complete normality, there exist open sets G and H such that $A \subset G$, $B \subset H$ and $G \cap H = \phi$. This implies that $(X, \mathbf{T})$ is normal. We know that, a T_5 space is T_1 also. Therefore $\mathbf{T}_5$ is both normal as well as $\mathbf{T}_1$. Hence every $\mathbf{T}_5$ space is $\mathbf{T}_4$.

Theorem 4: Complete normality is a topological property.

Proof: Consider a completely normal space $(X, \mathbf{T})$ and let $(Y, \mathbf{T}^*)$ be its homeomorphic image under a homeomorphism f. We have to prove that (Y, T^*) is completely normal. Let A and B be any two separated subsets of Y such that,

$$A \cap C(B) = \phi,\ B \cap C(A) = \phi$$

Now, as f is continuous mapping, therefore

$$C[f^{-1}(A)] \subset f^{-1}[C(A)] \text{ and } C[f^{-1}(B)] \subset f^{-1}[C(B)]$$

Hence, $f^{-1}(A) \cap C[f^{-1}(B)] \subset f^{-1}(A) \cap f^{-1}[C(B)]$

$$= f^{-1}[A \cap C(B)] = f^{-1}(\phi) = \phi$$

And

$$C[f^{-1}(A)] \cap f^{-1}(B) \subset f^{-1}[C(A)] \cap f^{-1}(B)$$

$$= f^{-1}[C(A) \cap B] = f^{-1}(\phi) = \phi$$

Thus $f^{-1}(A)$ and $f^{-1}(B)$ are two separated subsets of X. Since $(X, \mathbf{T})$ is completely normal, there exist $\mathbf{T}$-open sets G and H such that,

$$f^{-1}(A) \subset G,\ f^{-1}(B) \subset H \text{ and } G \cap H = \phi$$

So, we get that

$$A = f[f^{-1}(A)] \subset f(G)$$

$$B = f[f^{-1}(B)] \subset f(H) \quad \text{(Because of ontoness)}$$

$$\text{And } f(G) \cap f(H) = f(G \cap H) = f(\phi) = \phi \quad \text{(Because } f \text{ is 1–1)}$$

Also, since f is a open map, $f(G)$ and $f(H)$ are $\mathbf{T}^*$ open sets. Thus we have shown that for any two separated subsets A, B of Y, there exist $\mathbf{T}^*$ open subsets

$$G_1 = f(G) \text{ and } H_1 = f(H)$$

such that $A \subset G_1$, $B \subset H_1$ and $G_1 \cap H_1 = \phi$.

Hence the theorem is proved.

Corollary: The property of a space being $\mathbf{T}_5$ space is a topological property.

Proof: A $\mathbf{T}_5$ space is a completely normal $\mathbf{T}_1$ space. Since the properties of a space being a $\mathbf{T}_1$ space and of being a completely normal space, both are topological, therefore the property of a space being a $\mathbf{T}_5$ space is also topological.

Theorem 5: Complete normality is a hereditary property.

Proof: Suppose that $(X, \mathbf{T})$ is a completely normal space and $(Y, \mathbf{T}^*)$ is any subspace of $(X, \mathbf{T})$. We will be proving that $(Y, \mathbf{T}^*)$ is also completely normal. Let A, B be $\mathbf{T}^*$ separated subsets of $(Y, \mathbf{T}^*)$. We have,

$A \cap C^*(B) = \phi$ and $B \cap C^*(A) = \phi$

Also $C^*(A) = C(A) \cap Y$ and $C^*(B) = C(B) \cap Y$

Therefore,

$\phi = A \cap C^*(B) = A \cap [C(B) \cap Y] = (A \cap C(B)) \cap Y$

$= A \cap C(B)$

In the same way,

$\phi = B \cap C^*(A) = B \cap [C(A) \cap Y] = (B \cap C(A)) \cap Y$

$= B \cap C(A)$

So A and B are $\mathbf{T}$ separated. Hence by complete normality of X, there exist $\mathbf{T}$-open sets G and H such that,

$A \subset G$, $B \subset H$ and $G \cap H = \phi$

Since A and B are subsets of Y,

$A \subset G \cap Y$, $B \subset H \cap Y$

And $(G \cap Y) \cap (H \cap Y) = (G \cap H) \cap Y = \phi \cap Y = \phi$

Hence we can say that $(Y, \mathbf{T}^*)$ is completely normal.

Corollary: The property of a space being $\mathbf{T}_5$ space is a hereditary property.

Proof: Again, the property of a space being $\mathbf{T}_1$ as well as being completely normal is hereditary. Therefore, we can conclude that the property of a space being $\mathbf{T}_5$ is hereditary.

Theorem 6: A topological space X is completely normal if and only if every subspace of X is normal.

Proof: Consider a completely normal space $(X, \mathbf{T})$ and let $(Y < \mathbf{T}^*)$ be any subspace of $(X, \mathbf{T})$. We will show that $(Y, \mathbf{T}^*)$ is normal. Let F_1 and F_2 be any pair of T* closed subsets of Y such that $F_1 \cap F_2 = \phi$. We will denote the $\mathbf{T}^*$ closure of F_1 and F_2 by $C^*(F_1)$ and $C^*(F_2)$ and their $\mathbf{T}$ closure by $C(F_1)$ and $C(F_2)$, respectively.

NOTES

As F_1 and F_2 are T^* closed,

$C^*(F_1) = F_1$ and $C^*(F_2) = F_2$

Also, $C^*(F_1) = Y \cap C(F_1)$, $C^*(F_2) = Y \cap C(F_2)$

Therefore, $F_1 \cap C(F_2) = C^*(F_1) \cap C(F_2)$

$= (Y \cap C(F_1)) \cap C(F_2)$

$= (Y \cap C(F_1)) \cap (Y \cap C(F_2))$

$= C^*(F_1) \cap C^*(F_2)$

$= F_1 \cap F_2 = \phi$

In the same way, $F_2 \cap C(F_1) = \phi$.

We can now conclude that F_1 and F_2 are two separated subsets of X. From the definition of complete normality, there exist **T** open sets G and H such that $F_1 \subset G$, $F_2 \subset H$ and $G \cap H = \phi$. Then the sets $G_1 = Y \cap G$ and $H_1 = Y \cap H$ are T^* open sets such that,

$F_1 \subset G_1$, $F_2 \subset H_1$ and

$G_1 \cap H_1 = (Y \cap G) \cap (Y \cap H)$

$= Y \cap (G \cap H) = Y \cap \phi = \phi$

This implies that $(Y, \mathbf{T}^*)$ is normal.

Conversely suppose that $(Y, \mathbf{T}^*)$ is normal. Consider two separated subsets A and B of X.

Let $Y = X - (C(A) \cap C(B))$. Then Y is **T** open subset of X.

The sets $Y \cap C(A)$ and $Y \cap C(B)$ are $\mathbf{T}^*$ closed such that,

$(Y \cap C(A)) \cap (Y \cap C(B)) = Y \cap (C(A) \cap C(B))$

$= [X - (C(A) \cap C(B))] \cap [C(A) \cap C(B)]$

$= \phi$

Therefore by the normality of Y, there exist $\mathbf{T}^*$ open subsets G, H of Y such that

$$Y \cap C(A) \subset G,\ Y \cap C(B) \subset H \qquad \ldots(1)$$

And $G \cap H = \phi$

Again, since G and H are $\mathbf{T}^*$ open sets, there exist $\mathbf{T}^*$ open subsets G_1, H_1 of Y such that,

$G = Y \cap G_1$, $H = Y \cap H_1$

And since Y is **T**-open, we conclude that G and H are also **T**-open sets. Additionally, since

$Y = X–[C(A) \cap C(B)]$

Also, A and B are separated sets of X since

$Y = X–[C(A) \cap C(B)]$

$= [X–C(A)] \cup [X–C(B)]$

But, $X–C(A) \supset B$ and $X–C(B) \supset A$

$B \subset Y$ and $A \subset Y$

Hence it follows from Equation (1) that $A \subset G, B \subset H$ and $G \cap H = \phi$ ...(2)

Also A and B are separated since $B \subset X–C(A)$.

Thus we have proved that for any pair A, B of separated subsets of X, there exist **T** open subsets G and H which satisfy Equation (2). Therefore (X, **T**) is completely normal.

Note: Every completely normal space is normal but the converse may not be true. For example, consider

$X = \{a, b, c, d\}$

$T = \{\phi, (a), (a, c), (a, b, c), X\}$

Here (X, **T**) is normal since ϕ and X are the only disjoint closed subsets of X. But the subspace $Y = \{a, b, c\}$ of X is not normal. If **T*** is a relative topology in Y, then

$\mathbf{T}^* = \{\phi, \{a\}, \{a, b\}, \{a, c\}, Y\}$.

Here $\{b\}$ and $\{c\}$ are disjoint **T*** closed subsets of Y but there are no disjoint open sets G and H such that $\{b\} \subset G$ and $\{c\} \subset H$. Therefore the subspace (Y, **T***) is not normal. So we conclude that the space (X< **T**) is not completely normal

13.4.2 Completely Regular Space and $\mathbf{T}_{3\frac{1}{2}}$ Space

Definition: A topological space (X, **T**) is said to be completely regular iff it satisfies the axiom given as:

If F is a closed subset of X and x is a point of X not in F, then there exists a continuous map $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(x) = 1$.

Definition: A completely regular $\mathbf{T}_1$ space is known as a Tichonov space or $\mathbf{T}_{3\frac{1}{2}}$ space.

Theorem 7: Every completely regular space is regular and consequently every Tichonov space is a $\mathbf{T}_3$ space.

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Proof: Consider $(X, \mathbf{T})$ to be a completely regular space. Let F be a closed subset of X and let x be point of x not in F, i.e., $x \in X–F$. From the property of complete regularity, there exists a continuous map $f : X \to [0, 1]$ such that $f(X) = 0$ and $f(F) = \{1\}$. Also $[0, 1]$ with usual topology is a Hausdorff space. Therefore there exist open sets G and H of $[0, 1]$ such that,

$0 \in G$, $1 \in H$ and $G \cap H = \phi$.

Since f is continuous, $f^{-1}(G)$ and $f^{-1}(H)$ are $\mathbf{T}$-open subsets of X such that

$f^{-1}(G) \cap f^{-1}(H) = f^{-1}[G \cap H] = f^{-1}[\phi] = \phi$

Additionally,

$f(x) = 0 \in G$ implies $x \in f^{-1}(G)$

and $f(F) = \{1\} \in H$ implies $F \subset f^{-1}(H)$.

Thus there exist disjoint $\mathbf{T}$-open sets $f^{-1}(G)$ and $f^{-1}(H)$ that contain x and F, respectively. We infer that $(X, \mathbf{T})$ is regular. Moreover since every Tychonov space is both a completely regular and $\mathbf{T}_1$ space, we get that every Tychonov space is a $\mathbf{T}_3$ space.

Theorem 8: Every $\mathbf{T}_4$ space is a Tychonov space.

Proof: Consider (X, T) to be a $\mathbf{T}_4$ space. Then we can say that it is normal and $\mathbf{T}_1$ space. Hence we now only have to show that (X, T) is completely regular. Let F be a $\mathbf{T}$ closed subset of X and let x be a point of X such that $x \notin F$. Since the space is $\mathbf{T}_1$, $\{x\}$ is a closed subset of X. Thus $\{x\}$ and F are closed subsets of X. Again since the space is normal, by Urysohn's lemma, there exists a continuous mapping $f : X \to [0, 1]$ such that

$f[(x)] = \{0\}$ and $f(F) = \{1\}$

i.e., $f(x) = 0$ and $f(F) = 1$.

Corollary: Every compact Hausdorff space is a Tychonov space.

Proof: As every compact Hausdorff space is a normal space, hence it is also Tychonov space.

Theorem 9: Complete regularity is a topological property.

Proof: Let $(X, \mathbf{T})$ be a completely regular space and let (Y, V) be a homeomorphic image of $(X, \mathbf{T})$ under a homeomorphism f. To show that (Y, V) is completely regular, let F be a V-closed subset of Y and let y be a point of Y such that $y \notin F$. Since f is one-one onto, there exists a point $x \in X$ such that $f(x) = y \Leftrightarrow x = f^{-1}(Y)$. Again since f is a continuous map, $f^{-1}(F)$ is a $\mathbf{T}$-closed subset of X. Further $y \notin F \Rightarrow f^{-1}(y) \notin f^{-1}(F) \Rightarrow x \notin f^{-1}(F)$. Thus $f^{-1}(F)$ is a $\mathbf{T}$-closed subset of X and $x = f^{-1}(y)$ is a point of X such that $x \notin f^{-1}(F)$. Hence by complete regularity of X there exists a continuous mapping g of X into $[0, 1]$ such that

$g[f^{-1}(y)] = g(x) = 0$ and $g[f^{-1}(F)] = \{1\}$

i.e., $(gof^{-1})(y) = 0$ and $(gof^{-1})(F) = \{1\}$

Since f is homeomorphism, f^{-1} is a continuous mapping of Y onto X. Also g is a continuous map of X into [0, 1]. It follows that gof^{-1} is a continuous map of Y into [0, 1]. Here we have shown that for each V-closed subset F of Y and each point $y \in Y - F$, there exists a continuous map $h = gof^{-1}$ of Y into [0, 1] such that $h(y) = 0$ and $h(F) = \{1\}$. Hence we infer that (Y, V) is completely regular.

Theorem 10: Complete regularity is hereditary property.

Proof: Consider $(X, \mathbf{T})$ to be a completely regular space and let $(Y, \mathbf{T}^*)$ be a subspace of X. Let F^* be a closed subset of Y and y be a point of Y such that $y \notin F^*$. Since F^* is $\mathbf{T}^*$ closed, there exists a $\mathbf{T}$-closed subset F of X such that $F^* = Y \cap F$. Also,

$y \notin F^* \Rightarrow y \notin Y \cap F$

$\Rightarrow y \notin F$ [Since $y \in Y$]

And,

$y \in Y \Rightarrow y \in X$

This is a $\mathbf{T}$-closed subset of X and y is a point of X such that $y \notin F$. So by the property of complete regularity of X, there exists a continuous map f of X into [0, 1] such that

$f(y) = 0$ and $f(F) = \{1\}$

Let g denote the restriction of f to Y. Then g is continuous mapping of Y into [0, 1]. Now by definition of g,

$g(x) = f(x) \ \forall \ x \in Y$

Hence

$f(y) = 0, g(y) = 0$

and since $f(x) = 1 \ \forall \ x \in F$ and $F^* \subset F$, we have

$g(x) = f(x) = 1 \forall x \in F^*$

so that $g(F^*) = \{1\}$.

Here we have proved that for each $\mathbf{T}^*$ closed subset F^* of Y and each point $y \in Y - F^*$, there exists a continuous map g of Y into [0, 1] such that $g(y) = 0$ and $g(F^*) = \{1\}$.

Note: The property of a space of being a Tychonov space is both topological and hereditary property.

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Theorem 11: A normal space is completely regular iff it is regular.

Proof: Now, as every completely regular space is regular, we only need to show that any normal regular space is completely regular. Let F be a closed subset of X not containing the point x so that x belongs to the open set $X–F$. Since the space X is regular, there exists an open set G such that $x \in G$ and $C(G) \subset X–F$ so that

$$F \cap C(G) = \phi$$

Thus $C(G)$ and F are disjoint closed subsets of a normal space X. Hence by Urysohn's lemma, there exists a continuous mapping $f: X \to [0, 1]$ such that

$$f(F) = \{1\} \text{ and } f[C(G)] = \{0\}$$

Also $x \in G$ and $f[C(G)] = \{0\}$ implies that $f(x) = 0$. Therefore it follows that $(X, \mathbf{T})$ is completely regular.

Example 1: Consider a completely regular space $(X, \mathbf{T})$. Prove that if F is a $\mathbf{T}$-closed subset of X and $p \notin F$, then there exists a continuous mapping f of X into $[0, 1]$ such that $f(p) = 1$ and $f(F) = \{0\}$.

Since $(X, \mathbf{T})$ is completely regular, there exists a continuous map g of X into $[0, 1]$ such that

$$g(p) = 0 \text{ and } g(F) = \{1\}.$$

Now consider the mapping $f: X \to [0, 1]$ defined by setting $f(x) = 1–g(x)$ $\forall x \in X$.

Since constant functions are continuous, it follows that f is continuous.

Additionally, $\overline{f(p)} = 1–g(p) = 1–0 = 1$

$$f(x) = 1–g(x) = 1–1 \quad \forall x \in F$$
$$= 0$$

so that $f(F) = \{0\}$.

Example 2: Consider the topology $\mathbf{T}$ on $\mathbf{R}$ defined as: $\mathbf{T}$ consists of ϕ, $\mathbf{R}$ and all open sets of the form $[-\infty, a]$ such that $a \in \mathbf{R}$. Prove the following:

(*i*) $(\mathbf{R}, \mathbf{T})$ is normal

(*ii*) $(\mathbf{R}, \mathbf{T})$ is not regular

(*iii*) $(\mathbf{R}, \mathbf{T})$ is not $\mathbf{T}_4$

Solution: (*i*) Here the only disjoint closed subsets of $\mathbf{R}$ are of the form ϕ and $[a, \infty]$ and for each such pair, there exist disjoint open sets ϕ and R such that $\phi \subset \phi$ and $[a, \infty) \subset \mathbf{R}$. So we conclude that $(\mathbf{R}, \mathbf{T})$ is normal.

(*ii*) Consider the closed set $F = [1, \infty)$ and the point 0. Here $0 \notin F$. But the only open set containing F is $\mathbf{R}$ which must intersect every open set containing 0. Hence there exist no open sets G and H such that

$0 \in G$, $F \subset H$ and $G \cap H = \phi$.

This implies that the space (**R**, **T**) is not regular.

(*iii*) The space (**R**, **T**) is not $\mathbf{T}_1$ since no singleton subset of **R** is **T**-closed. Therefore the space is not $\mathbf{T}_4$.

Example 3: If x and y are two distinct points of a Tychonoff space (X, **T**), then there exists a real valued continuous mapping f of X such that $f(x) \neq f(y)$.

Solution: As (X, **T**) is a $\mathbf{T}_1$-space, the singleton subset $\{y\}$ is **T**-closed and since x and y are distinct points, $x \notin \{Y\}$. From the property of complete regularity of X, there exists a real valued continuous mapping f of X such that,

$f(x) = 0$ and $f[\{y\}] = \{1\}$

This implies,

$f(x) = 0$ and $f(y) = 1$

But $0 \neq 1$ implies $f(x) \neq f(y)$.

Notes: The following hierarchy exists:

a) Completely normality implies normality

b) Complete regularity implies regularity

c) Regularity implies Hausdorffness

d) Hausdorffness implies $\mathbf{T}_1$-ness

e) $\mathbf{T}_1$-ness implies $\mathbf{T}_0$-ness

Check Your Progress

5. State the Urysohn's lemma.
6. What is a completely normal topological space?

13.5 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1. A topological space X is a regular space if, given any nonempty closed set F and any point x, such that $x \notin F$, there exists a neighbourhood U of x and a neighbourhood V of F that are disjoint or we can say that it must be possible to separate x and F with disjoint neighbourhoods.
2. A topological space X is said to be a normal space if, given any disjoint closed sets E and F, there are open neighbourhoods U of E and V of F that are also disjoint.
3. A completely normal space or a hereditarily normal space is a topological space X in which every subspace of X with subspace topology is a normal space.

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4. A perfectly normal space is a topological space X in which every two disjoint non-empty closed sets E and F can be precisely separated by a continuous function f from X to the real line R.
5. Let X be a topological space and let any two disjoint closed sets A, B in X can be separated by open neighbourhoods. Then there is a continuous function $f: X \rightarrow [0, 1]$ such that $f | A \equiv 1$ and $f | B \equiv 0$.
6. A topological space (X, T) is called completely normal if and only if it satisfies the statement given as: if A and B are two separated subsets of X, then there exist two disjoint open sets G and H such that $A \subset G$ and $B \subset H$.

13.6 SUMMARY

- A topological space X is a regular space if, given any nonempty closed set F and any point x, such that $x \notin F$, there exists a neighbourhood U of x and a neighbourhood V of F that are disjoint or we can say that it must be possible to separate x and F with disjoint neighbourhoods.
- If a topological space is both regular and a Hausdorff space then it is said to be a T_3 space or regular Hausdorff space.
- A topological space is said to be topologically regular if every point has an open neighbourhood that is regular.
- A completely T_4 space or T_5 space is a completely normal Hausdorff topological space X, or every subspace of X must be a T_4 space.
- A locally normal space is a topological space where every point has an open neighbourhood that is normal.
- Let X be a topological space and let any two disjoint closed sets A, B in X can be separated by open neighbourhoods. Then there is a continuous function $f: X \rightarrow [0, 1]$ such that $f | A \equiv 1$ and $f | B \equiv 0$.
- If X is a normal topological space and A is closed in X, then for any continuous function $f: A \rightarrow R$ such that $|f(x) \leq 1$, there is a continuous function $g: X \rightarrow R$ such th

13.7 KEY WORDS

- **Regular space:** A topological space X is a regular space if, given any nonempty closed set F and any point x, such that $x \notin F$, there exists a neighbourhood U of x and a neighbourhood V of F that are disjoint or we can say that it must be possible to separate x and F with disjoint neighbourhoods.

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- **Normal space:** A topological space X is said to be a normal space if, given any disjoint closed sets E and F, there are open neighbourhoods U of E and V of F that are also disjoint.
- **Completely normal space:** A completely normal space or a hereditarily **normal space** is a topological space X in which every subspace of X with subspace topology is a normal space.
- **Perfectly normal space:** A perfectly normal space is a topological space X in which every two disjoint non-empty closed sets E and F can be precisely separated by a continuous function f from X to the real line **R**.
- **Completely regular space:** A topological space $(X, \mathbf{T})$ is said to be completely regular iff it satisfies the axiom given as:

  If F is a closed subset of X and x is a point of X not in F, then there exists a continuous map $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(x) = 1$.

13.8 SELF ASSESSMENT QUESTIONS AND EXERCISES

Short Answer Questions

1. Describe normal spaces briefly.
2. Write a short note on completely regular space.

Long Answer Questions

1. Describe the Urysohn's lemma.
2. Explain Tietze extension theorem.
3. Discuss complete regularity and normality.

13.9 FURTHER READINGS

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UNIT 14 THE URYSHON'S METRIZATION THEOREM

Structure

14.0 INTRODUCTION

One of the first widely recognized metrization theorems was Urysohn's metrization theorem. This states that every Hausdorff second countable regular space is metrizable. Every second-countable manifold is metrizable. Urysohn's Theorem can be restated as: A topological space is separable and metrizable if and only if it is regular, Hausdorff and second-countable. A space is said to be locally metrizable if every point has a metrizable neighbourhood. In this unit, you will discuss the Uryshon's metrization theorem, characterisation of the embedding topology, composition of embedding and characterisation of the product topology. You will be able to explain embedding lemma and Tychonoff's embedding theorem.

14.1 OBJECTIVES

After going through this unit, you will be able to:

- Discuss Uryshon's metrization theorem
- Understand characterisation of the embedding topology
- Analyse composition of embedding and characterisation of the product topology
- Describe embedding lemma and Tychonoff's embedding theorem

14.2 THE URYSHON'S METRIZATION AND EMBEDDING THEOREM

Determining if two given spaces are homeomorphic is one of the fundamental problems in topology.

Definition: A one-one and onto (bijection) continuous map $f : X \to Y$ is a homeomorphism if its inverse is continuous.

A bijection $f : X \to Y$ induces a bijection between subsets of X and subsets of Y and it is a homeomorphism iff this bijection restricts to a bijection,

$$\{\text{Open (or closed) subsets of } X\} \frac{U \to f(U)}{f^{-1}(V) \leftarrow V} \{\text{Open (or closed) subsets of } Y\}$$

between open (or closed) subsets of X and open (or closed) subsets of Y.

Definition: Suppose X is a set, Y a topological space and $f : X \to Y$ an injective map. The embedding topology on X (for the map f) is the collection,

$f^{-1}(\mathcal{T}_Y) = \{ f^{-1}(V) | V \subset Y \text{ open}\}$ of subsets of X.

The subspace topology for $A \subset X$ is the embedding topology for the inclusion map $A \to X$.

Theorem 1 (Characterization of the embedding topology): Let X has the embedding topology for the map $f : X \to Y$. Then,

1. $X \to Y$ is continuous.
2. For any map $A \to X$ into X,

   $A \to X$ is continuous iff $A \to X \xrightarrow{f} Y$ is continuous.

The embedding topology is the only topology on X with these two properties. The embedding topology is the most common topology on X such that $f : X \to Y$ is continuous.

Proof: The reason is that $A \xrightarrow{g} X$ is continuous.

$$\Leftrightarrow g^{-1}(\mathcal{T}_X) \subset \mathcal{T}_A \Leftrightarrow g^{-1}(f^{-1}\mathcal{T}_Y) \subset \mathcal{T}_A \Leftrightarrow (fg)^{-1}\,(\mathcal{T}_Y) \subset \mathcal{T}_A \Leftrightarrow A \xrightarrow{g} X \xrightarrow{f} Y$$

is continuous by definition of the embedding topology. The identity map of X is a homeomorphism whenever X is equipped with a topology with these two properties.

Definition: An injective continuous map $f : X \to Y$ is an embedding if the topology on X is the embedding topology for f, i.e., $\mathcal{T}_X = f^{-1}\mathcal{T}_Y$.

Any injective map $f : X \to Y$ induces a bijection between subsets of X and subsets of $f(X)$ and it is an embedding iff this bijection restricts to a bijection,

$$\{\text{Open (or closed) subsets of } X\} \frac{U \to f(U)}{f^{-1}(V) \leftarrow V} \{\text{Open (or closed) subsets of } f(X)\}$$

between open (or closed) subsets of X and open (or closed) subsets of $f(X)$.

Alternatively, the injective map $f : X \to Y$ is an embedding iff the bijective corestriction $f(X) | f : X \to f(X)$ is a homeomorphism. An embedding is a homeomorphism followed by an inclusion. The inclusion $A \to X$ of a subspace is an embedding. Any open (or closed) continuous injective map is an embedding.

For example, the map $f(x) = 3x + 1$ is a homeomorphism from $\mathbf{R} \to \mathbf{R}$.

Lemma: If $f: X \rightarrow Y$ is a homeomorphism (embedding) then the corestriction of the restriction $f(A)|f \backslash A: A \rightarrow f(A)$ $(B|f \backslash A: A \rightarrow B)$ is a homeomorphism (embedding) for any subset A of X (and any subset B of Y containing $f(A)$). If the maps $f_j: X_j \rightarrow Y_j$ are homeomorphisms (embeddings) then the product map $\prod f_j : \prod X_j \rightarrow Y_j$ is a homeomorphism (embedding).

Proof: In case of homeomorphisms employ that there is a continuous inverse in both cases. In case of embeddings, employ that an embedding is a homeomorphism followed by an inclusion map.

Lemma 2 (Composition of embeddings): Let $X \overset{f}{\rightarrow} Y \overset{g}{\rightarrow} Z$ be continuous maps. Then f and g are embeddings implies that $g \circ f$ is an embedding which in turn implies that f is an embedding.

Proof: For proving the second implication, first note that f is an injective continuous map. Let $U \subset X$ be open. Since $g \circ f$ is an embedding, $U = (g \circ f)^{-1}\ W$ for some open $W \subset Z$. But $(g \circ f)^{-1} = f^{-1}g^{-1}W$ where $g^{-1}W$ is open in Y since g is contimuous. This shows that f is an embedding.

Theorem 2 (Characterization of the product topology): Given the product topology $\prod Y_j$. Then,

1. The projections $\pi_j : \prod Y_j \rightarrow Y_j$ are continuous, and
2. For any map $f: X \rightarrow \prod_{j \in J} Y_j$ into the product space we have,

$$X \overset{f}{\rightarrow} \prod_{j \in J} Y_j \text{ is continuous} \Leftrightarrow \forall\ j \in J: X \overset{f}{\rightarrow} \prod_{j \in J} Y_j \overset{\pi_j}{\rightarrow} Y_j \text{ is continuous.}$$

The product topology is the only topology on the product set with these two properties.

Proof: Let $\mathbf{T}_X$ be the topology on X and $\mathbf{T}_j$ the topology on Y_j. Then $S_{\Pi} = \bigcup_{j \in J} \pi_j^{-1}(\mathcal{T}_j)$ is a subbasis for the product topology on $\prod_{j \in J} Y_j$. Therefore,

$$f: X \rightarrow \prod_{j \in J} Y_j \text{ is continuous} \Leftrightarrow f^{-1}(\bigcup_{j \in J} \pi_j^{-1}(\mathcal{T}_j)) \subset \mathcal{T}_X$$

$$\Leftrightarrow (\bigcup_{j \in J} f^{-1}(\pi_j^{-1}(\mathcal{T}_j))) \subset \mathrm{T}_X$$

$$\Leftrightarrow \forall\ j \in J: (\pi_j \circ f)^{-1}(\mathcal{T}_j) \subset \mathcal{T}_X$$

$$\Leftrightarrow \forall\ j \in J: \pi_j \circ f \text{ is continuous by definition of continuity}$$

Now, we have to show that the product topology is the unique topology with these properties. Take two copies of the product set $\prod_{j \in J} X_j$. Provide one copy with the product topology and the other copy with some topology that has the two properties of the above theorem. Then the identity map between these two copies is a homeomorphism.

Theorem 3: Let $(X_j)_{j\in J}$ be an indexed family of topological spaces with subspaces $A_j \subset X_j$. Then $\prod_{j\in J} A_j$ is a subspace of $\prod_{j\in J} X_j$.

1. $\overline{\prod A_j} = \prod \overline{A_j}$
2. $(\prod A_j)^\circ \subset \prod A_j^\circ$ and equality holds if $A_j = X_j$ for all but finitely many $j \in J$.

Proof: (1) Let (x_j) be a point of $\prod X_j$. Since $S_\Pi = \bigcup_{j\in J} \pi_j^{-1}(\mathcal{T}_j)$ is a subbasis for the product topology on $\prod X_j$, we have

$(x_j) \in \overline{\prod A_j} \Leftrightarrow \forall\, k \in J$: $\pi_k^{-1}(U_k) \cap \prod A_j \neq \varphi$ for all neighbourhoods U_k of x_k.

$\Leftrightarrow \forall\, k \in J$: $U_k \cap A_k \neq \phi$ for all neighbourhoods U_k of x_k

$\Leftrightarrow \forall\, k \in J$: $x_k \in \overline{A_k}$

$\Leftrightarrow (x_j) \prod \overline{A_j}$

(2) $(\prod A_j)^\circ \subset \prod A_j^\circ$ because π_j is an open map so that $\pi_j((\prod A_j)^\circ) \subset A_j^\circ$ for all $j \in J$. If $A_j = X_j$ for all but finitely many $j \in J$ then $\prod A_j^\circ \subset (\prod A_j)^\circ$ because $\prod A_j^\circ$ is open and contained in $\prod A_j$.

It follows that a product of closed sets is closed.

Note: A product of open sets need not be open in the product topology.

14.2.1 Embedding Lemma and Tychonoff Embedding

Theorem 4 (Embedding lemma): Let $\mathcal{F}$ be a family of mappings where each member $f \in \mathcal{F}$ maps $X \rightarrow Y_f$. Then,

1. The evaluation mapping $e: X \rightarrow \prod_{f\in\mathcal{F}} Y_f$ defined by $\pi_f oe(x) = f(x)$, for all $x \in X$, is continuous.
2. The mapping e is an open mapping onto $e(X)$ if $\mathcal{F}$ distinguishes points and closed sets.
3. The mapping e is one-to-one if and only if $\mathcal{F}$ distinguishes points.
4. The mappping e is an embedding if $\mathcal{F}$ distinguished points F distinguishes points and closed sets.

Proof: (1) Let $\pi_g: \prod_{f\in\mathcal{F}} Y_f \rightarrow Y_g$ be the projection map to the space Y_g. Then $\pi_g oe = g$ so that $\pi_g oe$ is continuous. Therefore e must be continuous as g is continuous.

(2) Suppose that U is open in X and $x \in U$. Choose $f \in \mathcal{F}$ such that $f(x) \notin \overline{f(X \setminus U)}$. The set $B = \{z \in e(X) | \pi_f(z) \notin \overline{f(X \setminus U)}\}$ is a neighbourhood of $e(x)$ as the set is open (it is defined for components not being in the closed set $\overline{f(X \setminus U)}$ and clearly $e(x) \in B$. Moreover $\pi_f(B) \subset f(U)$ by construction. It is now claimed that $f(U) \subset \pi_f(B)$. This follows trivially from the definition of a family of functions distinguishing points and closed sets. Therefore $f(U) = \pi_f(B)$ and subsequently $f(U)$ is an open subset of $\pi_g o e(X)$. Therefore the evaluation map is an open mapping.

(3) The definition of distinguishing points implies injectivity.

(4) Combining a, b and c, we see that $X \cong e(X)$ as e is a continuous, open, injective, surjective (as a continuous map is always surjective onto its image) map.

Definition: If X is a space and A, a set then by the power X^A we mean the product space $\Pi_\alpha X_\alpha$, where $X_\alpha = X$, for each $\alpha \in A$. Any power of [0, 1] is called a cube. A map e: $X \to Y$ is an embedding iff the map e: $X \to e(X)$ is a homeomorphism. If there is an embedding e: $X \to Y$ then we say that X can be embedded in Y.

Theorem 5 (Tychonoff's embedding theorem): A space is Tychonoff iff it can be embedded in a cube.

Proof: $\Rightarrow$ Let X be a Tychonoff space and let $A = \{f : X \to [0, 1] / f \text{ is continuous}\}$. Define e: $X \to [0, 1]^A$ by $e(x)(f) = f(x)$.

(*i*) e is injective: If $x, y \in X$ with $x \neq y$, then there is $f \in A$ so that $f(x) = 0$ and $f(y) = 1$. Then $e(x)(f) \neq e(y)(f)$, so $e(x) \neq e(y)$.

(*ii*) e is continuous: This is immediate since $\pi_f e = f$.

(*iii*) e: $X \to e(X)$ carries open sets of X to open subsets of $e(X)$: For let U be open in X and let $x \in U$. Then there is $f \in A$ so that $f(x) = 0$ and $f(X - U) = 1$. Let $V = \pi_f^{-1}([0, 1))$, an open subset of $[0, 1]^A$. Then $e(x) \in V$ and if $y \in X$ is such that $e(y) \in V$, then $e(y)(f) \in [0, 1)$, so $f(y) < 1$ and $y \in U$. Thus $e(x) \in V \cap e(X) \subset e(U)$.

(*i*), (*ii*) and (*iii*) together imply that e is an embedding.

$\Leftarrow$: [0, 1] is clearly so $[0, 1]^A$ is Tychonoff for any A. Any subspace of a Tychonoff space is Tychonoff. Thus if X can be embedded in a cube, then X is homeomorphic to a Tychonoff space and so is itself Tychonoff.

Theorem 6: Let $(\mathbf{T}, \mathcal{T})$ be the 3-point topological space defined by $\mathbf{T} = \{0, 1, 2\}$ and $\mathcal{T} = \{\varphi, \{0\}, \mathbf{T}\}$. Let $(X, \mathcal{U})$ be any topogical space and suppose that $\mathcal{U} \cap X = \varphi$. Then there is an embedding e: $X \to \mathcal{T}^{\mathcal{U} \cup X}$.

Proof: For each $U \in \mathcal{U}$, define $f_U: X \to \mathbf{T}$ by $f_U(y) = 0$ if $y \in U$ and $f_U(y)=1$ if $y \notin U$. Then f_U is continuous. For each $x \notin X$, define $f_x: X \to \mathbf{T}$ by $f_x(y) = 2$ if $y = x$ and $f_x(y) = 1$ if $y \neq x$. Then f_x is also continuous.

Define e by $e_i(y)=f_i(y)$ for each $i \in \mathcal{U} \cup X$. Then

(*i*) e is injective, for if $x, y \in X$ with $x \neq y$ then $e_x(y) = 1$ but $e_x(x) = 2$, so $e_x(x) \neq e_x(y)$ and hence $e(x) \neq e(y)$.

(*ii*) e is continuous because each f_i is continuous.

(*iii*) e is open into $e(X)$, for if $U \in \mathcal{U}$ and $x \in U$ then $V = \pi_U^{-1}(0)$ is open in $\mathbf{T}^{\mathcal{U} \cup X}$. Furthermore, so $\pi_U e(x)=0$, so $e(x) \in V$ while if $y \in X$ is such that $e(y) \in V$ then $\pi_U e(y)= 0$ and hence $y \in U$. Thus $V \cap e(X) \subset e(\mathcal{U})$.

14.2.2 Urysohn's Metrization Theorem

Theorem 7 (Urysohn's metrization theorem): Suppose $(X, \mathcal{T})$ is a regular topological space with a countable basis $\mathcal{B}$, then X is metrizable.

Proof: Let $(X, \mathcal{T})$ be a regular metrizable space with countable basis $\mathcal{B}$. For this proof, we will first create a countable collection of functions $\{f_n\}_{n\in\mathbf{N}}$, where f_m:$X \to \mathbf{R}$ for all $m \in \mathbf{N}$, such that given any $x \in X$ and any open neighbourhood U of x there is an index N such that $f_N(x) > 0$ and zero outside of U. We will then use these functions to imbed X in $\mathbf{R}^w$.

Let $x \in X$ and let U be any open neighbourhood of x. There exists $B_m \in \mathcal{B}$ such that $x \in B_m$. Now, since X has a countable basis and is regular, we know that X is normal. Next, as B_m is open, there exists some $B_n \in \mathcal{B}$ such that $\overline{B_n} \subset B_m$. Thus we now have two closed sets $\overline{B_n}$ and $X \backslash B_m$, and so we can apply Urysohn's lemma to give us a continuous function $g_{n,m}: X \to \mathbf{R}$ such that $g_{n,m}(\overline{B_n}) = \{1\}$ and $g_{n,m}(X \backslash B_m) = \{0\}$. Notice here that this function satisfies requisite: $g_{n,m}(y) = 0$ for $y \in X \backslash B_m$ and $g_{n,m}(x) > 0$. Here, g was indexed purposely, as it shows us that $\{g_{n,m}\}$ is indexed by $\mathbf{N}\times\mathbf{N}$, which is countable (since the cross product of two countable sets is countable). Considering this, relable the functions $\{g_{n,m}\}_{n,m\in\mathbf{N}}$ as $\{f_n\}_{n\in\mathbf{N}}$.

We now imbed X in the metrizable space $\mathbf{R}^w$. Let $F: X \to \mathbf{R}^w$ where $F(x) = (f_1(x), f_2(x), f_3(x), \ldots)$, where f_n are the functions constructed above. We claim that F is an imbedding of X into $\mathbf{R}^w$.

For F to be an imbedding it is required of F to be homeomorphic onto its image. First, this needs that F should be a continuous bijection onto its image. We know that F is continuous as each of its component functions f_N are continuous by construction. Now we show that F is an injection.

Let $x, y \in X$ be distinct. From the Hausdorff condition there exist open sets U_x and U_y such that $x \in U_x$, $y \in U_y$ with $U_x \cap U_y = \phi$. By the construction of our

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maps f there exists an index $N \in \mathbf{N}$ such that $f_N(U_x) > 0$ and $F_N(X \setminus U_x) = 0$. It follows that $f_N(x) \neq f_N(y)$ and so $F(x) \neq F(y)$. Hence, F is injective.

Now, as it is clear that F is surjective onto its image $F(X)$, all that is left to show is that F is an embedding. We will show that for any open set $U \in X$, $F(U)$ is open in $\mathbf{R}^w$. Let $U \subset X$ be open and let $x \in U$. Pick an index N such that $f_N(x) > 0$ and $f_N(X \setminus U) = 0$. Let $F(x) = z \in F(U)$. Let $V = \pi_N^{-1}((0, \infty))$, i.e., all elements of $\mathbf{R}^w$ with a positive Nth coordinate. Now let $W = F(X) \cap V$. We claim that $z \in W \subset F(U)$ showing that $F(U)$ can be written as a union of open sets, hence making it open.

First we show that W is open in $F(X)$. We know that V is an open set in $\mathbf{R}^w$. $W = F(X) \cap V$, and W is open by the definition of the subspace topology.

Thereafter, we will first show that $f(x) = z \in W$ and then $W \subset F(U)$. To prove our first claim, $F(x) = z \Rightarrow (F(x)) = f_N(x) > 0 \Rightarrow -\pi_N(z) = \pi_N(F(x)) = f_N(x) > 0 \Rightarrow \pi_N(z) > 0$ which means that $z \in \pi_N^{-1}(V)$ and also $z \in F(X) \Rightarrow z \in F(X) \cap V = W$. Now we show that $W \subset F(U)$. Let $y \in W$. This means $y \in F(X) \cap V = W$.

Now we show that $W \subset F(U)$. Let $y \in W$. This means $y \in F(X) \cap V$. This means there exists some $w \in X$ such that $F(w) = y$. But, since $y \in V$ we have that:

$\pi_N(y) = \pi_N(F(w)) = f_N(w) > 0$ since $y \in V$, but $f_N(w) = 0$ for all $w \in X \setminus U$ and so $y \in F(U)$.

In conclusion, as we have shown that $F: X \rightarrow \mathbf{R}^w$ is a map that preserves open sets in both directions and bijective onto its image, we have shown that F is an embedding of the space X into the metrizable space $\mathbf{R}^w$ and X is therefore metrizable, the metric being given by the induced metric from $\mathbf{R}^w$.

Example 1: The topology generated by the dictionary ordering on $\mathbf{R}^2$ is metrizable.

Proof: From previous Theorem 3.19, all we have to do for showing that $\mathbf{R}^2$ is metrizable in the dictionary ordering is to prove that this space is regular with a countable basis.

Now, since the set $\{(a, b), (c, d) | a \leq c, b < d; a, b, c, d \in \mathbf{R}\}$ is a basis for the dictionary ordering on $\mathbf{R}^2$ and the set of intervals with rational end-points are a basis for the usual topology on $\mathbf{R}$, it follows that the set $\{(a, b), (c, d) | a \leq c, b < d; a, b, c, d \in \mathbf{Q}\}$ is a countable basis for the dictionary ordering.

Now we will show that the dictionary ordering is regular. Let $a \in \mathbf{R}^2$ and $B \subseteq \mathbf{R}^2$ such that B is closed in the dictionary ordering and $a \notin B$. Let $\varepsilon = \inf\{d(a, b) | b \in B\}$. We know that ε is greater than 0, for otherwise a would be an accumulation point of B, which is a contradiction. It follows that the open sets $((a, a-\varepsilon/2), (a, a+\varepsilon/2))$ and $\bigcup_{b \in B} ((b, b-\varepsilon/2), (b, b+\varepsilon/2))$ are disjoint open sets containing a and B, respectively. Hence, the dictionary ordering over $\mathbf{R}^2$ is metrizable, since it is regular and has a countable basis.

Note: In this proof we have shown that a sequence of functions $\{f_n\}_{n\in\mathbf{N}}$ with the property that for each $x \in X$ and each neighbourhood U of x there is some $n \in \mathbf{N}$ such that $f_n(x) > 0$ and $f_n(y) > 0$ for all $y \in X\backslash U$, gives us an imbedding $F: X \to \mathbf{R}^w$. Notice that we have the very similar result if we have a sequence of functions $\{f_j\}_{j\in J}$ with same properties as above: given any $x \in X$ and any neighbourhood U of x there exists $j \in J$ such that $f_j(x) > 0$ and $f_j(y) = 0$ for all $y \in X \backslash U$, then we have an imbedding from $X \to \mathbf{R}^J$ given by $F(x) = (f_j(x))_{j\in J}$. This is known as the **imbedding theorem** and is a generalization of Urysohn's metrization theorem.

Check Your Progress

1. Define homeomorphism.
2. Give definition of embedding.
3. State Tychonoff's embedding theorem.
4. What is the Urysohn's metrization theorem?
5. When is a map an embedding?

14.3 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1. A one-one and onto (bijection) continuous map $f: X \to Y$ is a homeomorphism if its inverse is continuous.
2. An injective continuous map $f: X \to Y$ is an embedding if the topology on X is the embedding topology for f, i.e., $\mathcal{T}_X = f^{-1}\mathcal{T}_Y$.
3. A space is Tychonoff iff it can be embedded in a cube.
4. Suppose $(X, \mathcal{T})$ is a regular topological space with a countable basis $\mathcal{B}$, then X is metrizable.
5. A map $e: X \to Y$ is an embedding iff the map $e: X \to e(X)$ is a homeomorphism. If there is an embedding $e: X \to Y$ then we say that X can be embedded in Y.

14.4 SUMMARY

- A one-one and onto (bijection) continuous map $f: X \to Y$ is a homeomorphism if its inverse is continuous.
- Suppose X is a set, Y a topological space and $f: X \to Y$ an injective map. The embedding topology on X (for the map f) is the collection, $f^{-1}(\mathcal{T}_Y) = \{f^{-1}(V) | V \subset Y \text{ open}\}$ of subsets of X.
- The injective map $f: X \to Y$ is an embedding iff the bijective corestriction $f(X)|f: X \to f(X)$ is a homeomorphism.
- If $f: X \to Y$ is a homeomorphism (embedding) then the corestriction of the restriction $f(A)|f \backslash A: A \to f(A)$ $(B|f \backslash A: A \to B)$ is a homeomorphism (embedding) for any subset A of X (and any subset B of Y containing $f(A)$).

- A space is Tychonoff iff it can be embedded in a cube.
- Suppose $(X, \mathcal{T})$ is a regular topological space with a countable basis $\mathcal{B}$, then X is metrizable.

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14.5 KEY WORDS

- **Tychonoff's embedding theorem:** A space is Tychonoff iff it can be embedded in a cube.
- **Urysohn's metrization theorem:** Suppose (X, T) is a regular topological space with a countable basis B, then X is metrizable.

14.6 SELF ASSESSMENT QUESTIONS AND EXERCISES

Short Answer Questions

1. Write a short note on embedding lemma and Tychonoff's embedding.
2. Describe characterisation of the embedding topology.
3. Explain characterisation of the product topology.

Long Answer Questions

1. Discuss the Urysohn's embedding theorem.
2. Describe the Urysohn's metrization theorem.

14.7 FURTHER READINGS

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